STAT743 FOUNDATIONS OF STATISTICS (PART II)

Winter 2019

Assignment 3 Solutions

Q. 1 **a)** Since the sample come from a normal population with variance σ^2, we know that $T = (n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$ so

$$\begin{aligned} \mathrm{E}(cS) &= \frac{c\sigma}{\sqrt{n-1}}\mathrm{E}\big(\sqrt{T}\big) \\ &= \frac{c\sigma}{\sqrt{n-1}}\int_0^\infty \frac{\sqrt{t}}{\Gamma((n-1)/2)2^{(n-1)/2}} t^{\frac{n-1}{2}-1}\mathrm{e}^{-\frac{t}{2}}\,dt \\ &= \frac{c\sigma\Gamma(n/2)\sqrt{2}}{\Gamma((n-1)/2)\sqrt{n-1}}\int_0^\infty \frac{1}{\Gamma(n/2)2^{n/2}} t^{\frac{n}{2}-1}\mathrm{e}^{-\frac{t}{2}}\,dt \\ &= \frac{c\Gamma(n/2)\sqrt{2}}{\Gamma((n-1)/2)\sqrt{n-1}}\sigma \end{aligned}$$

Hence we should take

$$c = \frac{\Gamma((n-1)/2)\sqrt{n-1}}{\Gamma(n/2)\sqrt{2}}.$$

[10 marks]

b) As suggested I will work with $Y_1, \ldots, Y_n$ where $Y_i = X_i - \mu$. Note that

$$\overline{Y} = \overline{X} - \mu \qquad S^2_Y = S^2_X$$

and $\mathrm{E}(\overline{Y}) = 0$ so

$$\begin{aligned} \mathrm{Cov}\left(\overline{X}, S^2_X\right) &= \mathrm{Cov}(\overline{Y}, S^2_Y) \\ &= \mathrm{E}\big(\overline{Y}S^2_Y\big) \\ &= \frac{1}{n-1}\mathrm{E}\left(\overline{Y}\left(\sum_{j=1}^n Y_j^2 - n\overline{Y}^2\right)\right) \\ &= \frac{1}{n-1}\left\{\mathrm{E}\left(\overline{Y}\sum_{j=1}^n Y_j^2\right) - n\mathrm{E}\left(\overline{Y}^3\right)\right\} \end{aligned}$$

[4 marks]

Now the first of these expectations is

$$\begin{aligned}
\mathrm{E}\left(\overline{Y}\sum_{j=1}^{n}Y_j^2\right) &= \frac{1}{n}\mathrm{E}\left(\left(\sum_{i=1}^{n}Y_i\right)\left(\sum_{j=1}^{n}Y_j^2\right)\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left(Y_iY_j^2\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(Y_i^3\right)+\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}\mathrm{E}\left(Y_i\right)\mathrm{E}\left(Y_j^2\right)\\
&\qquad\text{(because } Y_i \text{ and } Y_j \text{ are independent for } i\neq j\text{)}\\
&= \mathrm{E}\left(Y^3\right)\qquad\text{(because } \mathrm{E}\left(Y_i\right)=0\text{)}
\end{aligned}$$

[4 marks]

The second we get similarly

$$\begin{aligned}
\mathrm{E}\left(\overline{Y}^3\right) &= \frac{1}{n^3}\mathrm{E}\left(\left(\sum_{i=1}^{n}Y_i\right)\left(\sum_{j=1}^{n}Y_j\right)\left(\sum_{k=1}^{n}Y_k\right)\right)\\
&= \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathrm{E}\left(Y_iY_jY_k\right)\\
&= \frac{1}{n^3}\sum_{i=1}^{n}\mathrm{E}\left(Y_i^3\right)+\frac{3}{n^3}\sum_{i=1}^{n}\sum_{j\neq i}\mathrm{E}\left(Y_iY_j^2\right)+\frac{1}{n^3}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\notin\{i,j\}}\mathrm{E}\left(Y_iY_jY_k\right)\\
&= \frac{1}{n^3}\sum_{i=1}^{n}\mathrm{E}\left(Y_i^3\right)+\frac{3}{n^3}\sum_{i=1}^{n}\sum_{j\neq i}\mathrm{E}\left(Y_i\right)\mathrm{E}\left(Y_j^2\right)+\frac{1}{n^3}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\notin\{i,j\}}\mathrm{E}\left(Y_i\right)\mathrm{E}\left(Y_j\right)\mathrm{E}\left(Y_k\right)\\
&= \frac{1}{n^2}\mathrm{E}\left(Y^3\right)
\end{aligned}$$

[5 marks]

Hence we have

$$\begin{aligned}
\mathrm{Cov}\left(\overline{X},S_X^2\right) &= \frac{1}{n-1}\left\{\mathrm{E}\left(Y^3\right)-\frac{n}{n^2}\mathrm{E}\left(Y^3\right)\right\}\\
&= \frac{1}{n}\mathrm{E}\left(Y^3\right)\\
&= \frac{1}{n}\mathrm{E}\left((X-\mu)^3\right)
\end{aligned}$$

[2 marks]

Q. 2 **a)** Let $X_1 \sim \chi^2_p$ and $X_2 \sim \chi^2_q$ be independent random variables. Then their joint pdf is given by

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\Gamma(p/2)\Gamma(q/2)2^{(p+q)/2}} x_1^{p/2-1} x_2^{q/2-1} \mathrm{e}^{-(x_1+x_2)/2}$$

Consider the transformation

$$\begin{aligned} u &= \frac{x_1/p}{x_2/q} \\ v &= \frac{x_2}{q} \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= puv \\ x_2 &= qv \end{aligned}$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} pv & pu \\ 0 & q \end{vmatrix} = pqv$$

and the support is

$$\left.\begin{aligned} 0 < x_1 < \infty \\ 0 < x_2 < \infty \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} 0 < puv < \infty \\ 0 < qv < \infty \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} 0 < u < \infty, \\ 0 < v < \infty \end{aligned}\right.$$

[4 marks]

Hence the joint density of (U,V) is

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{\Gamma(p/2)\Gamma(q/2)2^{(p+q)/2}} (puv)^{p/2-1}(qv)^{q/2-1}\mathrm{e}^{-(puv+qv)/2}|pqv| \qquad \begin{aligned} 0 < u < \infty \\ 0 < v < \infty \end{aligned} \\ &= \frac{p^{p/2}q^{q/2}}{\Gamma(p/2)\Gamma(q/2)2^{(p+q)/2}} u^{p/2-1} v^{(p+q)/2-1} \exp\left\{-\frac{v(pu+q)}{2}\right\} \qquad \begin{aligned} 0 < u < \infty \\ 0 < v < \infty \end{aligned} \end{aligned}$$

[2 marks]

We now integrate to get the marginal density for U

$$\begin{aligned} f_U(u) &= \int_0^\infty \frac{p^{p/2}q^{q/2}}{\Gamma(p/2)\Gamma(q/2)2^{(p+q)/2}} u^{p/2-1} v^{(p+q)/2-1} \exp\{-\frac{v(pu+q)}{2}\}\, dv \\ &= \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \frac{p^{p/2}q^{q/2}u^{p/2-1}}{(pu+q)^{(p+q)/2}} \int_0^\infty \frac{\left(\frac{pu+q}{2}\right)^{(p+q)/2}}{\Gamma\left(\frac{p+q}{2}\right)} v^{\frac{p+q}{2}-1} \exp\left\{-\left(\frac{pu+q}{2}\right)v\right\} dv \\ &= \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \frac{p^{p/2}q^{q/2}u^{p/2-1}}{(pu+q)^{(p+q)/2}} \\ &= \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \left(\frac{p}{q}\right)^{p/2} \frac{u^{p/2-1}}{\left(1+\frac{p}{q}u\right)^{(p+q)/2}} \qquad \text{for } 0 < u < \infty \end{aligned}$$

[4 marks]

b) Let $Y \sim F_{p,q}$ then we can write

$$Y \stackrel{d}{=} \frac{X_1/p}{X_2/q}$$

where $X_1 \sim \chi^2_p$ and $X_2 \sim \chi^2_q$ are independent random variables so we have

$$\begin{aligned} \mathrm{E}(Y) &= \frac{q}{p}\mathrm{E}\left(X_1\right)\mathrm{E}\left(X_2^{-1}\right) \\ \mathrm{E}\left(Y^2\right) &= \frac{q^2}{p^2}\mathrm{E}\left(X_1^2\right)\mathrm{E}\left(X_2^{-2}\right) \end{aligned}$$

From the information on Page 623 we know that

$$\mathrm{E}\left(X_1\right) = p \qquad \mathrm{E}\left(X_1^2\right) = 2p + p^2$$

so we only need to find the moments of the reciprocals of a chi-squared random variable.

$$\begin{aligned} \mathrm{E}\left(X_2^{-r}\right) &= \int_0^\infty \frac{1}{\Gamma(q/2)2^{q/2}} x^{q/2-r-1}\mathrm{e}^{-x/2}\,dx \\ &= \frac{\Gamma\left(\frac{q-2r}{2}\right)}{\Gamma\left(\frac{q}{2}\right)2^r}\int_0^\infty \frac{1}{\Gamma\left(\frac{q-2r}{2}\right)2^{(q-2r)/2}} x^{(q-2r)/2-1}\mathrm{e}^{-x/2}\,dx \\ &= \frac{\Gamma\left(\frac{q-2r}{2}\right)}{\Gamma\left(\frac{q}{2}\right)2^r} \qquad \text{provided } r < \frac{q}{2} \end{aligned}$$

Hence we have

$$\begin{aligned} \mathrm{E}\left(X_2^{-1}\right) &= \frac{\Gamma\left(\frac{q}{2}-1\right)}{2\Gamma\left(\frac{q}{2}\right)} = \frac{1}{q-2} && \text{provided } q > 2 \\ \mathrm{E}\left(X_2^{-2}\right) &= \frac{\Gamma\left(\frac{q}{2}-2\right)}{4\Gamma\left(\frac{q}{2}\right)} = \frac{1}{(q-2)(q-4)} && \text{provided } q > 4 \end{aligned}$$

[5 marks]

Thus, the moments of the F distribution are

$$\begin{aligned} \mathrm{E}\left(Y\right) &= \frac{q}{p} \times p \times \frac{1}{q-2} = \frac{q}{q-2} && \text{provided } q > 2 \\ \mathrm{E}\left(Y^2\right) &= \frac{q^2}{p^2} \times (2p - p^2) \times \frac{1}{(q-2)(q-4)} \\ &= \frac{q^2(p+2)}{p(q-2)(q-4)} && \text{provided } q > 4 \\ \mathrm{Var}\left(Y\right) &= \frac{q^2(p+2)}{p(q-2)(q-4)} - \frac{q^2}{(q-2)^2} \\ &= \frac{q^2}{q-2}\left[\frac{p+2}{p(q-4)} - \frac{1}{q-2}\right] \\ &= \frac{2q^2(p+q-2)}{p(q-2)^2(q-4)} && \text{provided } q > 4 \end{aligned}$$

[4 marks]

c) Suppose that $X \sim F_{p,q}$ and let

$$Y = \frac{(p/q)X}{1+(p/q)X}$$

The inverse of this transformation is

$$x = \frac{qy}{p(1-y)} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{q}{p(1-y)^2}$$

and the support of the distribution is fond by

$$0 < x < \infty \quad \Rightarrow \quad 0 < \frac{pX}{q} < 1 + \frac{pX}{q} < \infty\infty \quad \Rightarrow \quad 0 < \frac{(p/q)X}{1+(p/q)X} < 1$$

[3 marks]

Using the result of part (a) for the density of X and Theorem 22 in my notes, the density function for Y is

$$\begin{aligned}
f_Y(y) &= f_X\left(\frac{qy}{p(1-y)}\right)\left|\frac{q}{p(1-y)^2}\right| \\
&= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{p}{q}\right)^{p/2}\frac{\left(\frac{qy}{p(1-y)}\right)^{p/2-1}}{\left(1+\frac{y}{1-y}\right)^{(p+q)/2}}\left|\frac{q}{p(1-y)^2}\right| \\
&= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\frac{\left(\frac{y}{1-y}\right)^{p/2-1}}{\left(\frac{1}{1-y}\right)^{(p+q)/2}(1-y)^2} \\
&= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}y^{p/2-1}(1-y)^{q/2-1}
\end{aligned}$$

We now recognize this density as that of a Beta random variable with parameters $\alpha = p/2$ and $\beta = q/2$.

[3 marks]

Q. 3 **a)** For convenience I will use X_r to denote a random variable having a χ^2_r distribution. Now suppose that X_q and X_{p-q} are independent Chi-squared random variables. Then we have, from textbook Lemma 5.3.2, that $X_q + X_{p-q} \sim \chi^2_p$.

Hence, for any $a > 0$ we have

$$\begin{aligned} \mathrm{P}(X_p > a) &= \mathrm{P}(X_q + X_{p-q} > a) \\ &= \mathrm{P}(X_q > a - X_{p-q}) \\ &> \mathrm{P}(X_q > a \mid X_{p-q} > 0)\mathrm{P}(X_{p-q} > 0) \\ &= \mathrm{P}(X_q > a)\mathrm{P}(X_{p-q} > 0) && \text{independence of } X_q \text{ and } X_{p-q} \\ &= \mathrm{P}(X_q > a) && \text{since } \mathrm{P}(X_{p-q} > 0) = 1 \end{aligned}$$

[5 marks]

b) In parts (b) and (c) of this question I will let $\phi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function and cumulative distribution function of the standard normal respectively.

Let $Y = \min(Z_1, Z_2)$. To show that Y does not have a standard normal distribution it is only necessary to show that

$$\mathrm{P}(Y \leqslant a) \neq \Phi(a)$$

for at least one $a \in \mathbb{R}$. It is easiest to take $a = 0$ which I will do here.

$$\begin{aligned} \mathrm{P}(Y \leqslant 0) &= 1 - \mathrm{P}(Y > 0) \\ &= 1 - \mathrm{P}(Z_1 > 0,\ Z_2 > 0) \\ &= 1 - \mathrm{P}(Z_1 > 0)\mathrm{P}(Z_2 > 0) && \text{by independence of } Z_1 \text{ and } Z_2 \\ &= 1 - 0.5 \times 0.5 \\ &= 0.75 \\ &\neq \Phi(0) \end{aligned}$$

Hence Y is not standard normal. **[4 marks]**

Now consider the cumulative distribution function of Y^2.

$$\begin{aligned}
F_{Y^2}(y) &= \mathrm{P}(Y^2 \leqslant y) \\
&= \mathrm{P}(-\sqrt{y} \leqslant Y \leqslant \sqrt{y}) \\
&= \mathrm{P}(Y \leqslant \sqrt{y}) - \mathrm{P}(Y < -\sqrt{y}) \\
&= \big(1 - \mathrm{P}(Y > \sqrt{y})\big) - \big(1 - \mathrm{P}(Y \geqslant -\sqrt{y})\big) \\
&= \mathrm{P}(Y \geqslant -\sqrt{y}) - \mathrm{P}(Y > \sqrt{y}) \\
&= \mathrm{P}(Z_1 \geqslant -\sqrt{y},\ Z_2 \geqslant -\sqrt{y}) - \mathrm{P}(Z_1 > -\sqrt{y},\ Z_2 > -\sqrt{y}) \\
&= \mathrm{P}(Z_1 \geqslant -\sqrt{y})\mathrm{P}(Z_2 \geqslant -\sqrt{y}) - \mathrm{P}(Z_1 > -\sqrt{y})\mathrm{P}(Z_2 > -\sqrt{y}) \\
&= \big(1 - \Phi(-\sqrt{y})\big)^2 - \big(1 - \Phi(\sqrt{y})\big)^2 \\
&= \big(\Phi(\sqrt{y})\big)^2 - \big(1 - \Phi(\sqrt{y})\big)^2 \qquad \text{by symmetry of the standard normal} \\
&= 2\Phi(\sqrt{y}) - 1
\end{aligned}$$

[6 marks]

Hence the probability density function of Y^2 is

$$\begin{aligned}
f_{Y^2}(y) = \frac{d}{dy}F_{Y^2}(y) &= \frac{d}{dy}\big(2\Phi(\sqrt{y}) - 1\big) \\
&= 2\phi(\sqrt{y})\frac{1}{2\sqrt{y}} \\
&= \frac{1}{\sqrt{2\pi y}}e^{-y/2} \\
&= \frac{1}{\Gamma(0.5)2^{0.5}}y^{0.5-1}\mathrm{e}^{-y/2}
\end{aligned}$$

which we recognize as the pdf of the gamma($\alpha = 0.5, \beta = 2$) distribution and that is the χ^2_1 distribution so we have that $Y^2 \sim \chi^2_1$. **[4 marks]**

c) If we now define $Y = \min(Z_1, \ldots, Z_n)$ where $Z1, \ldots, Z_n$ are independent standard normal random variables then by the same process as in part (b) we have that the cdf of Y^2 is

$$\begin{aligned}
F_{Y^2}(y) &= \mathrm{P}(-\sqrt{y} \leqslant Y \leqslant \sqrt{y}) \\
&= \mathrm{P}(Y \geqslant -\sqrt{y}) - \mathrm{P}(Y > \sqrt{y}) \\
&= \mathrm{P}(Z_i \geqslant -\sqrt{y},\ i = 1, \ldots, n) - \mathrm{P}(Z_i > -\sqrt{y},\ i = 1, \ldots, n) \\
&= \mathrm{P}(Z_1 \geqslant -\sqrt{y})\mathrm{P}(Z_2 \geqslant -\sqrt{y}) - \mathrm{P}(Z_1 > -\sqrt{y})\mathrm{P}(Z_2 > -\sqrt{y}) \\
&= \big(\Phi(\sqrt{y})\big)^n - \big(1 - \Phi(\sqrt{y})\big)^n \\
&\neq 2\Phi(\sqrt{y}) - 1 \qquad \text{unless } n = 2
\end{aligned}$$

Since we saw in part (b) that the cdf of the χ^2_1 distribution is $2\Phi(\sqrt{y}) - 1$, we see that Y^2 does not have a χ^2_1 distribution if $n > 2$. **[6 marks]**

Q. 4 **a)** First note that

$$\begin{aligned} \mathrm{P}\left(X_{(n)} \leqslant v\right) &= \mathrm{P}\left(X_{(1)} \leqslant u, X_{(n)} \leqslant v\right) + \mathrm{P}\left(X_{(1)} > u, X_{(n)} \leqslant v\right) \\ &= F_{X_{(1)},X_{(n)}}(u,v) + \mathrm{P}\left(X_{(1)} > u, X_{(n)} \leqslant v\right) \end{aligned}$$

[2 marks]

Now we have

$$\mathrm{P}\left(X_{(n)} \leqslant v\right) = \mathrm{P}\left(X_i \leqslant v, i = 1, \ldots, n\right) = \prod_{i=1}^{n} \mathrm{P}(X_i \leqslant v) = \left(F_X(v)\right)^n$$

Furthermore we see that

$$\begin{aligned} \mathrm{P}\left(X_{(1)} > u, X_{(n)} \leqslant v\right) &= \mathrm{P}\left(u < X_i \leqslant v, i = 1, \ldots, n\right) \\ &= \prod_{i=1}^{n} \mathrm{P}\left(u < X_i \leqslant v\right) \qquad \text{(by independence of } X_1, \ldots, X_n\text{)} \\ &= \left(F_X(v) - F_X(u)\right)^n \end{aligned}$$

provided $u < v$ and the probability is 0 otherwise. **[2 marks]**

Hence the joint cdf of $X_{(1)}$ and $X_{(n)}$ is

$$F_{X_{(1)},X_{(n)}}(u,v) = \begin{cases} \left(F_X(v)\right)^n - \left(F_X(v) - F_X(u)\right)^n & \text{if } u < v \\ \left(F_X(v)\right)^n & \text{if } u \geqslant v \end{cases}$$

[2 marks]

We can get the joint pdf by taking derivatives of this to get

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(u,v) &= \frac{\partial^2}{\partial u \partial v} F_{X_{(1)},X_{(n)}}(u,v) \\ &= \begin{cases} \dfrac{\partial}{\partial v}\left\{ n f_X(u) \left(F_X(v) - F_X(u)\right)^{n-1} \right\} & \text{if } u < v \\ 0 & \text{if } u \geqslant v \end{cases} \\ &= \begin{cases} n(n-1) f_X(u) f_X(v) \left(F_X(v) - F_X(u)\right)^{n-2} & \text{if } u < v \\ 0 & \text{if } u \geqslant v \end{cases} \end{aligned}$$

[2 marks]

b) For the exponential(θ) I will use the form in your textbook but either standard parameterization is acceptable. Then from the result of in part (a) we have that the joint density of $(X_{(1)}, X_{(n)})$ is

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(x_1, x_n) &= n(n-1)\left(\frac{1}{\theta}\mathrm{e}^{-x/\theta}\right)\left(\frac{1}{\theta}\mathrm{e}^{-x/\theta}\right)\left(\mathrm{e}^{-x_1/\theta} - \mathrm{e}^{-x_n/\theta}\right)^{n-2} \\ &= \frac{n(n-1)}{\theta^2}\mathrm{e}^{-(x_1+x_n)/\theta}\left(\mathrm{e}^{-x_1/\theta} - \mathrm{e}^{-x_n/\theta}\right)^{n-2} \qquad \text{for } 0 < x_1 < x_n < \infty \end{aligned}$$

[2 marks]

Now let us consider the transformation $W = X_{(n)} - X_{(1)}$ and $U = X_{(1)}$

$$\begin{aligned} w &= x_n - x_1 \\ u &= x_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= u \\ x_n &= u + w \end{aligned} \quad \Rightarrow \quad J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

The support of the transformation is found from

$$0 < x_1 < x_n < \infty \quad \Rightarrow \quad 0 < u < u + w < \infty \quad \Rightarrow \quad \begin{cases} 0 < u < \infty \\ 0 < w < \infty \end{cases}$$

[2 marks]

Hence the joint density of (U, W) for $u > 0$ and $w > 0$ is

$$\begin{aligned} f_{W,U}(w, u) &= f_{X_{(1)},X_{(n)}}(u, u + w) \\ &= \frac{n(n-1)}{\theta^2}\mathrm{e}^{-(2u+w)/\theta}\left(\mathrm{e}^{-u/\theta} - \mathrm{e}^{-(u+w)/\theta}\right)^{n-2} \times |1| \\ &= \frac{n(n-1)}{\theta^2}\mathrm{e}^{-2u/\theta}\mathrm{e}^{-w/\theta}\mathrm{e}^{-(n-2)u/\theta}\left(1 - \mathrm{e}^{-w/\theta}\right)^{n-2} \\ &= \left(\frac{n}{\theta}\mathrm{e}^{-nu/\theta}\right)\left(\frac{n-1}{\theta}\mathrm{e}^{-w/\theta}\left(1 - \mathrm{e}^{-w/\theta}\right)^{n-2}\right) \end{aligned}$$

[2 marks]

Since the joint density factors into separate functions of u and w for all $(u, w) \in \mathbb{R}^2$ we see that $W = X_{(n)} - X_{(1)}$ and $U = X_{(1)}$ are independent random variables. **[1 mark]**

From the second part of Theorem 6.15 we know that the marginal density of $X_{(1)}$ is

$$f_{X_{(1)}}(u) \;=\; nf_X(u)\big[1-F_X(u)\big]^{n-1} \;=\; n\left(\frac{1}{\theta}\mathrm{e}^{-u/\theta}\right)\left(\mathrm{e}^{-u/\theta}\right)^{n-1} \;=\; \frac{n}{\theta}\mathrm{e}^{-nu/\theta} \qquad \text{for } u>0$$

[1 mark]

From this and the independence of W and U we get

$$f_W(w) \;=\; \frac{f_{W,U}(w,u)}{f_U(u)} \;=\; \frac{n-1}{\theta}\mathrm{e}^{-w/\theta}\left(1-\mathrm{e}^{-w/\theta}\right)^{n-2} \qquad \text{for } w>0$$

[1 mark]

c) **Casella and Berger 5.24**

From the first part of the question we have

$$f_{X_{(1)},X_{(n)}}(x_1,x_n) \;=\; n(n-1)\frac{1}{\theta^2}\left(\frac{x_n}{\theta}-\frac{x_1}{\theta}\right)^{n-2} = \frac{n(n-1)(x_n-x_1)^{n-2}}{\theta^n} \qquad \text{for } 0<u<v<\theta$$

[1 mark]

Now we consider the bivariate transformation $W = X_{(1)}/X_{(n)}$, $U = X_{(n)}$ from which we get

$$\begin{array}{lcl} w &=& \dfrac{x_1}{x_n} \\ u &=& x_n \end{array} \;\Rightarrow\; \begin{array}{lcl} x_1 &=& uw \\ x_n &=& u \end{array} \;\Rightarrow\; J = \begin{vmatrix} w & u \\ 1 & 0 \end{vmatrix} = -u$$

The support of the distribution is found from

$$0<x_1<x_n<\theta \quad\Rightarrow\quad 0<uw<u<\theta \quad\Rightarrow\quad \begin{cases} 0<u<\theta \\ 0<w<1 \end{cases}$$

[2 marks]

Hence the joint density of (U,W) is

$$\begin{aligned} f_{U,W}(u,w) &= f_{X_{(1)},X_{(n)}}(uw,u)|J| \\ &= \frac{n(n-1)(u-uw)^{n-2}}{\theta^n}|-u| \qquad \text{for } 0<uw<u<\theta \\ &= \frac{n(n-1)}{\theta^n}u^{n-1}(1-w)^{n-2} \qquad \text{for } 0<u<\theta,\; 0<w<1 \end{aligned}$$

[2 marks]

Since this joint density factors into a function of u alone and a function of w alone, the random variables $U = X_{(n)}$ and $W = X_{(1)}/X_{(n)}$ are independent. **[1 mark]**

From Theorem 6.18 we have that the cdf of X_n is

$$F_{X_{(n)}}(x) \;=\; \left(\frac{x}{\theta}\right)^n \qquad \text{for } 0 < x < \theta$$

and so the density of the maximum is

$$f_{X_{(n)}}(x) \;=\; \frac{nx^{n-1}}{\theta^n} \qquad \text{for } 0 < x < \theta$$

From the independence of $X_{(n)}$ and $X_{(1)}/X_{(n)}$ we then have that the density of $W = X_{(1)}/X_{(n)}$ is

$$\begin{aligned} f_W(w) &= \frac{f_{W,X_{(1)}}(w,x)}{f_{X_{(1)}}(x)} \\ &= \frac{\left(\frac{n(n-1)}{\theta^n}x^{n-1}(1-w)^{n-2}\right)}{\left(\frac{nx^{n-1}}{\theta^n}\right)} \\ &= (n-1)(1-w)^{n-2} \qquad 0 < w < 1 \end{aligned}$$

So we see that $W = X_{(1)}/X_{(n)} \sim \text{beta}(1, n-1)$. **[2 marks]**