Introduction to Probability and Basic Statistical Inference

Qihao Xie

Department of Mathematics and Statistics
McMaster University

STATISTICS 2D03, FALL 2009

Outline

- Combinatorial Analysis (§ 1.1 - 1.5)
- Axioms of Probability (§ 2.1 - 2.5, and 2.7)
- Conditional Probability and Independence (§ 3.1 - 3.5)
- Discrete Random Variables and Discrete Distributions (§ 4.1 - 4.9)
- Continuous Random Variables and Continuous Distributions (§ 5.1 - 5.6)
- Transformations (§ 5.7)
- Joint Distributed Random Variables (§ 6.1 - 6.5)
- Properties of Expectation (§ 7.1, 7.2, 7.4, and 7.5)
- Moment Generating Function (§ 7.7)
- Central Limit Theorems (§ 8.3)

Introduction to Probability Theory

In this section, we will cover the following topics:

- Combinatorial Analysis (§ 1.2 - 1.5)
- Axioms of Probability (§ 2.2 - 2.5 & 2.7)

◊ What is Probability?

- Whether we have math skills or not, we frequently estimate and compare probabilities, sometimes without realizing it, especially when making decisions.
- Beyond the word *probability*, there is a whole conception developed by Probability Theory.
- Any of its interpretation cannot leave aside the mathematical notation.
- Probability
  Probability is defined to be a measure of how frequently the event will occur or has occurred from a random experiment.
- Random Experiment
  A *Random Experiment* is an experiment that can be repeated numerous times under the same condition and result in different outcomes, for which the outcome cannot be determined in advance.

◊ What is Probability?

- Examples
  - Rolling a die - 6 possible outcomes with number 1 to 6 as the die faces are labeled
  - Tossing a coin - 2 possible outcomes, heads or tails
  - Selecting a numbered ball (1-20) in an urn - 20 possible outcomes
  - Drawn in a lottery

◊ Combinatorial Analysis

- In fact, many problems in probability theory can be involved simply by counting the number of different ways that a certain event can occur.
- **Combinatorial Analysis**
  The *Combinatorial Analysis* is the mathematical theory of counting that deals with permutations and combinations, especially used in probability and statistics.
- **Basic Principal of Counting**
  Suppose 2 experiments are to be performed. If experiment 1 can result in any one of m possible outcomes, and if for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.
  Proof:

◊ Combinatorial Analysis

- **Example 1.1**

  How many 2 letter words can be formed from the letter a, b, c and d, if repetition are not allowed and any sequence of 2 letter is defined to be a word?

  Solution:

- **Generalized Basic Principal of Counting**

  Suppose r experiments are to be performed and that experiment i may result any of n_i possible outcomes, for $i = 1, \ldots, r$. Then, there is a total number of $n_1 \cdot n_2 \cdots n_r$ distinct possible outcomes of the r experiments.

◊ Combinatorial Analysis

- **Example 1.2**

  In Example 1.1 above, how many 3 letter words can be obtained?

  Solution:

- **Example 1.3**

  How many different ordered arrangements of the letters a, b and c are possible?

  Solution:

◊ Combinatorial Analysis

- **Permutations**
  The number of distinct permutations (or arrangements) of n objects of which n_i is alike in the i^{th} group, for $i = 1, \ldots, r$, is given by

$$\frac{n!}{n_1! \cdot n_2! \cdots n_r!},$$

  where $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$.

  Proof: (using the Generalized Basic Principal of Counting)

◊ Combinatorial Analysis

- **Note 1.1:** The number of different permutations of n distinct objects from a set of n without repetition is $n!$.

- **Note 1.2:** The number of different permutations of r distinct objects from a set of n $(r \leq n)$ without repetition is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!} .$$

Proof:

◊ Combinatorial Analysis

- **Example 1.4**
  How many licence plate can be made if a licence plate consists of exactly 10 numbers ($0-9$ digits) without repetition?
  Solution:

- **Example 1.5**
  How many licence plate can be obtained if a licence plate consists of exactly 6 numbers selected from $0-9$ without repetition?
  Solution:

- **Example 1.6**
  How many licence plate can be obtained using all of three 0's, two 1's and two 2's?
  Solution:

◊ Combinatorial Analysis

- **Combinations**

Suppose we want to select a group of r items from a set of n $(r \leq n)$ items.

According to GBPC, there are r experiments being counted when the order of selection is relevant.

Recalled from *Note 1.2*, the number of different ways that a group of r items could be selected from n items when the order of selected is important is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!} \ .$$

However, each group consisting of r items will be counted $r!$ times. It follows that the number of different groups of r items that can be formed from a set of n items is given by

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \ .$$

◊ Combinatorial Analysis

- **Notation**
  We define "n" choose "r" by

$$\binom{n}{r} = \begin{cases} \frac{n!}{(n-r)!r!}, & 0 \leq r \leq n \\ 0, & r < 0 \text{ or } r > n \end{cases}$$

  with convention $0! = 1$.

  Notice that

$$0! = 1 \implies \binom{n}{0} = \binom{n}{n} = 1.$$

- **Note 1.3:** $\binom{n}{r}$ represents the number of different groups of size r that can be selected from a set of n objects when the order of selection is **not** consider relevant.

◊ Combinatorial Analysis

- Recalled from *Note 1.2*, when order is relevant, the number of different groups of size r that can be selected is $\frac{n!}{(n-r)!}$.

  This includes $r!$ ordering for each selection of r.

  Thus, when order is **not** relevant, the number of different group is

$$\frac{n!}{(n-r)!} \Big/ r! = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

◊ Combinatorial Analysis

- **Notation**
  We define the *multinomial coefficient* by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! \cdot n_2! \cdots n_r!},$$

  where $n_1 + n_2 + \cdots + n_r = n$.

- **Note 1.4:** $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

◊ Combinatorial Analysis

- **Example 1.7**
  In how many ways can a committee of 4 people be formed from a group of 10 people?
  Solution:

- **Example 1.8**
  How man 5 cards poker hands can be dealt?
  Solution:

- **Example 1.9**
  A class of 60 students will be writing a test in 3 rooms that will be hold 15, 20, and 25, respectively. In how many different ways can the 60 students be assigned to the 3 rooms?
  Solution:

◊ Combinatorial Analysis

- **Combinatorial Identity**

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad 1 \leq r \leq n.$$

Proof:

- Notice that the values $\binom{n}{r}$ are often referred as *binomial coefficients*.
- **Binomial Theorem**

$$(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k}.$$

Proof: (by induction)

◊ Combinatorial Analysis

- **Multinomial Theorem**

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1,\ldots,n_r)\,:\, n_1+\cdots+n_r = n} \binom{n}{n_1,\ldots,n_r} x_1^{n_1} \cdot x_2^{n_2} \cdots x_r^{n_r}.$$

Proof: (by induction)