Chapt.12: Orthogonal Functions and Fourier series

J.-P. Gabardo

gabardo@mcmaster.ca

Department of Mathematics & Statistics
McMaster University
Hamilton, ON, Canada
12.1: Orthogonal Functions

- Recall that $\mathbb{R}^n = \{(x_1, \ldots, x_n), \ x_i \in \mathbb{R}, \ i = 1, \ldots n\}$.

- If $u = (u_1 \ldots, u_n)$ and $v = (v_1 \ldots, v_n)$ belong to $\mathbb{R}^n$, their **dot product** is the number

$$ (u, v) = u_1 v_1 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i. $$

- The dot product has the following properties:
  - $(u, v) = (v, u)$
  - $(\alpha u, v) = \alpha (u, v) = (u, \alpha v), \ \alpha \in \mathbb{R}$
  - $(u + v, w) = (u, w) + (v, w)$
  - $(u, u) \geq 0$ and $(u, u) = 0$ iff $u = 0$
Orthogonal collections

- The norm of a vector: \( \| \mathbf{u} \| = \sqrt{u_1^2 + \cdots + u_n^2} = (\mathbf{u}, \mathbf{u})^{1/2} \)
- Orthogonality of two vectors: \( \mathbf{u} \perp \mathbf{v} \) iff \( (\mathbf{u}, \mathbf{v}) = 0 \).
- Orthogonality of a collection of vectors: \( \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \) is an orthogonal collection of vectors iff \( (\mathbf{u}_i, \mathbf{u}_j) = 0 \) if \( i \neq j \).
- Orthogonal basis: If \( m = n \), the dimension of the space, then an orthogonal collection \( \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \) where \( \mathbf{u}_i \neq 0 \) for all \( i \), forms an orthogonal basis. In that case, any vector \( \mathbf{v} \in \mathbb{R}^n \) can be expanded in terms of the orthogonal basis via the formula

\[
\mathbf{v} = \sum_{i=1}^{n} (\mathbf{v}, \mathbf{u}_i) \frac{\mathbf{u}_i}{\| \mathbf{u}_i \|^2}.
\]

- Orthonormal basis: orthogonal basis \( \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \) with \( \| \mathbf{u}_i \| = 1 \) for all \( i \).
Orthogonal Functions

- In what follows, we will always assume that the functions considered are piecewise continuous on some interval \([a, b]\).
- **Inner product**: If \(f_1, f_2\) are two functions defined on \([a, b]\), we define their inner product as

\[
(f_1, f_2) = \int_{a}^{b} f_1(x) f_2(x) \, dx
\]

- **Orthogonality**: Two functions \(f_1, f_2\) are orthogonal on \([a, b]\) if \((f_1, f_2) = 0\).
- **Example**: \(f(x) = \sin(3x), \, g(x) = \cos(3x)\). We have

\[
\int_{-\pi}^{\pi} \sin(3x) \cos(3x) \, dx = 0
\]

since \(\sin(3x) \cos(3x)\) is odd and the interval \([-\pi, \pi]\) is symmetric about 0. Thus \(f(x) = \sin(3x)\) and \(g(x) = \cos(3x)\) are orthogonal on \([-\pi, \pi]\).
Orthogonal Functions contd.

• **Example:** $f(x) = \sin(3x)$, $g(x) = \cos(3x)$. We have

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• **Orthogonal collections:** A collection of functions

$
\{\phi_0(x), \phi_1(x), \ldots, \phi_m(x), \ldots\}$ defined on $[a, b]$ is called orthogonal on $[a, b]$ if

$$(\phi_i, \phi_j) = \int_{a}^{b} \phi_i(x) \phi_j(x) \, dx = 0, \text{ when } i \neq j.$$
An example

The collection \( \{1, \cos(x), \cos(2x), \cos(3x), \ldots \} = \{\cos(kx), k \geq 0\} \) is orthogonal on \([-\pi, \pi]\).

- To show this, we use the identity

\[
\cos A \cos B = \frac{\cos(A + B) + \cos(A - B)}{2}.
\]

- We have, if \(m, n \geq 0\) are integers with \(m \neq n\),

\[
\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx \\
= \int_{-\pi}^{\pi} \frac{\cos((m + n)x) + \cos((m - n)x)}{2} \, dx \\
= \frac{1}{2} \left[ \frac{\sin((m + n)x)}{m + n} + \frac{\sin((m - n)x)}{m - n} \right]_{-\pi}^{\pi} = 0
\]
Orthonormality

- If $f(x)$ is a function defined on $[a, b]$, we define the norm of $f$ to be

$$
\|f\| = (f, f)^{1/2} = \left( \int_{a}^{b} f(x)^{2} \, dx \right)^{1/2}
$$

- A collection of functions $\{\phi_0(x), \phi_1(x), \ldots, \phi_m(x), \ldots\}$ defined on $[a, b]$ is called orthonormal on $[a, b]$ if

$$
(\phi_i, \phi_j) = \int_{a}^{b} \phi_i(x) \phi_j(x) \, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}
$$

- Note that if the collection $\{\phi_0(x), \phi_1(x), \ldots, \phi_m(x), \ldots\}$ is orthogonal on $[a, b]$ and $\|\phi_i\| \neq 0$, the collection $\left\{ \frac{\phi_0(x)}{\|\phi_0\|}, \frac{\phi_1(x)}{\|\phi_1\|}, \ldots, \frac{\phi_m(x)}{\|\phi_m\|}, \ldots \right\}$ is orthonormal on $[a, b]$. 
An example

• Consider the collection \{1, \cos(x), \cos(2x), \cos(3x), \ldots \} or 
\{\cos(kx), k \geq 0\} which we have shown to be is orthogonal on 
\([-\pi, \pi]\).

• We have \|1\|^2 = \int_{-\pi}^{\pi} 1^2 \, dx = 2\pi, so \|1\| = \sqrt{2\pi}.

• For \(m \geq 1\), we have

\[\|\cos(mx)\|^2 = \int_{-\pi}^{\pi} \cos^2(mx) \, dx = \int_{-\pi}^{\pi} \frac{1 + \cos(2mx)}{2} \, dx\]

\[= \left[ \frac{x}{2} + \frac{\sin(2mx)}{4m} \right]_{-\pi}^{\pi} = \pi.\]

• Thus \(\|\cos(mx)\| = \sqrt{\pi}\).

• The collection \(\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\cos(3x)}{\sqrt{\pi}}, \ldots \right\}\) is thus orthonormal 
on \([-\pi, \pi]\).
Section 12.1 continued

• Suppose that the collection \( \{ \phi_n(x) \}_{n \geq 0} \) is an orthogonal collection (or “system”) on \([a, b]\) and that the function \( f(x) \) defined on \([a, b]\) can be expanded as a series

\[
f(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + \cdots + c_n \phi_n(x) + \ldots,
\]

how can we compute the coefficients \( c_0, c_1, c_2, \ldots \)?

• Note that if (1) holds, we have, for each \( n \geq 0 \),

\[
(f, \phi_n) = \int_a^b f(x) \phi_n(x) \, dx = \int_a^b \left\{ \sum_{k=0}^{\infty} c_k \phi_k(x) \right\} \phi_n(x) \, dx
\]

\[
= \sum_{k=0}^{\infty} c_k \int_a^b \phi_k(x) \phi_n(x) \, dx = c_n \int_a^b \phi_n^2(x) \, dx = c_n \| \phi_n \|^2.
\]
Orthogonal systems

- It follows thus that, if (1) holds, then

\[ c_n = \frac{(f, \phi_n)}{\|\phi_n\|^2}, \quad n \geq 0. \]

and

\[ f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n\|^2} \phi_n(x). \quad (2) \]

- However, the expansion formula (2) does not hold in general for an arbitrary orthogonal system on \([a, b]\). For example, it could happen that \(f \neq 0\) but \(f(x)\) is orthogonal to each function \(\phi_n(x)\) in the system and thus the RHS of (2) would be 0 in that case while \(f(x) \neq 0\).

- In order for (2) to hold for an arbitrary function \(f(x)\) defined on \([a, b]\), there must be “enough” functions \(\phi_n\) in our system.
Completeness

- **Definition:** An orthogonal system \( \{ \phi_n(x) \}_{n \geq 0} \) on \([a, b]\) is **complete** if the fact that a function \( f(x) \) on \([a, b]\) satisfies \( (f, \phi_n) = 0 \) for all \( n \geq 0 \) implies that \( f \equiv 0 \) on \([a, b]\), or, more precisely, that \[
\|f\|^2 = \int_a^b f^2(x) \, dx = 0.
\]

- If \( \{ \phi_n(x) \}_{n \geq 0} \) on \([a, b]\) is a complete orthogonal system on \([a, b]\), then every (piecewise continuous) function \( f(x) \) on \([a, b]\) has the expansion

\[
f(x) \simeq \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n\|^2} \phi_n(x). \tag{3}
\]

on \([a, b]\) in the \(L^2\)-sense which means that

\[
\lim_{N \to \infty} \int_a^b \left| f(x) - \sum_{n=0}^{N} \frac{(f, \phi_n)}{\|\phi_n\|^2} \phi_n(x) \right|^2 \, dx = 0.
\]
Some remarks

- If \( \{ \phi_n(x) \}_{n \geq 0} \) on \([a, b]\) is a complete orthogonal system on \([a, b]\), the expansion formula (3) holds for every (pwc) function \( f(x) \) on \([a, b]\) in the \(L^2\)-sense, but not necessarily “pointwise”, i.e. for a fixed \( x \in [a, b] \) the series on the RHS of (3) might not necessarily converge and, even if it does, it might not converge to \( f(x) \).

- The system \( \{1, \cos(x), \cos(2x), \cos(3x), \ldots \} = \{\cos(kx), k \geq 0\} \) is orthogonal on \([-\pi, \pi]\) but it is not complete on \([-\pi, \pi]\).

- Indeed, if \( f(x) \) any odd function on \([-\pi, \pi]\) \((f(-x) = -f(x))\) with \( \|f\| \neq 0 \), such as \( f(x) = x \) or \( f(x) = \sin x \), we have

\[
\int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = 0, \quad n \geq 0,
\]

since \( f(x) \cos(nx) \) is odd and \([-\pi, \pi]\) is symmetric about 0.
Section 12.2: Fourier series

• **Theorem:** The system

\[ \mathcal{T} := \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \ldots \} \]

is a **complete orthogonal system** on \([-\pi, \pi]\).

• To show the orthogonality of this system, one needs to show that

\[
\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = 0, \quad m, n \geq 0, m \neq n, \quad (a)
\]

\[
\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = 0, \quad m, n \geq 1, m \neq n, \quad (b)
\]

\[
\int_{-\pi}^{\pi} \cos(mx) \sin(nx) \, dx = 0, \quad m \geq 0, n \geq 1. \quad (c)
\]
Fourier series contd.

- For example, to show (b), we use the formula

\[
\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}.
\]

- We have then, if \( m, n \geq 1 \) and \( m \neq n \),

\[
\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx
= \int_{-\pi}^{\pi} \frac{\cos((m - n)x) - \cos((m + n)x)}{2} \, dx
= \left[ \frac{\sin((m - n)x)}{m - n} - \frac{\sin((m + n)x)}{m + n} \right]_{-\pi}^{\pi} = 0.
\]

- We have also, for \( m, n \geq 1 \),

\[
\|1\|^2 = 2\pi, \quad \|\cos(mx)\|^2 = \pi, \quad \|\sin(nx)\|^2 = \pi.
\]
Fourier series expansions

- Note that the completeness of the system $\mathcal{T}$ is much more difficult to prove.

- Using the previous theorem, it follows that every (pwc) function $f(x)$ on $[-\pi, \pi]$ admits the expansion

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\} \quad (4),$$

where $\frac{a_0}{2} = \frac{(f, 1)}{\|1\|^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \text{average of } f \text{ on } [-\pi, \pi]$,

$$a_n = \frac{(f, \cos(nx))}{\|\cos(nx)\|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \quad n \geq 1,$$

$$b_n = \frac{(f, \sin(nx))}{\|\sin(nx)\|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx, \quad n \geq 1.$$
Fourier series on general intervals

- The series expansion (4) in terms of the trigonometric system \( \mathcal{T} \) is called the Fourier series expansion of \( f(x) \) on \([-\pi, \pi]\).

- More generally, if \( p > 0 \) and \( f(x) \) is pwc on \([-p, p]\), then it will have a Fourier series expansion on \([-p, p]\) given by

\[
f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \left( \frac{n\pi x}{p} \right) + b_n \sin \left( \frac{n\pi x}{p} \right) \right\}
\]

where the Fourier coefficients \( a_n, b_n \) are defined by

\[
a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \left( \frac{n\pi x}{p} \right) \, dx, \quad n \geq 0,
\]

\[
b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \left( \frac{n\pi x}{p} \right) \, dx, \quad n \geq 1.
\]
An example

- The function

\[ f(x) = \begin{cases} 
0, & -\pi < x \leq 0 \\
x, & 0 < x < \pi 
\end{cases} \]

has a Fourier series expansion on \([-\pi, \pi]\) given by

\[
f(x) \simeq \frac{\pi}{4} + \sum_{n \geq 1, \text{n odd}} \left( \frac{-2}{\pi n^2} \right) \cos(nx) + \sum_{n \geq 1} \left( \frac{-1}{n} \right)^{n+1} \sin(nx) \\
\simeq \frac{\pi}{4} - \frac{2}{\pi} \cos(x) - \frac{2}{9\pi} \cos(3x) - \frac{2}{25\pi} \cos(5x) + \ldots \\
\sin(x) - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) - \frac{1}{4} \sin(4x) + \ldots \quad (*)
\]
Periodic extension

• If a function $f(x)$ defined on the interval $[-p, p]$ is expanded as the Fourier series

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \left( \frac{n\pi x}{p} \right) + b_n \sin \left( \frac{n\pi x}{p} \right) \right\} \quad (5),$$

we can view the RHS of (5) as a function defined on all of $\mathbb{R}$.

• Since

$$\cos \left( \frac{n\pi (x + 2p)}{p} \right) = \cos \left( \frac{n\pi x}{p} + 2n\pi \right) = \cos \left( \frac{n\pi x}{p} \right),$$

$$\sin \left( \frac{n\pi (x + 2p)}{p} \right) = \sin \left( \frac{n\pi x}{p} + 2n\pi \right) = \sin \left( \frac{n\pi x}{p} \right),$$

the RHS of (5) is $2p$-periodic and thus equal to the $2p$-periodic extension of $f(x)$ to the real line.
Piecewise continuity

- Recall that a function $f(x)$ defined on the interval $[a, b]$ is piecewise continuous (pwc) on $[a, b]$ if $[a, b]$ can be divided into $N$ subintervals $[a_i, a_{i+1}]$, $i = 0, \ldots, N - 1$ with $a = a_0 < a_1 < a_2 < \cdots < a_{N-1} < a_N = b$ and such that $f(x)$ is continuous on each open interval $(a_i, a_{i+1})$, $i = 0, \ldots, N - 1$ and

\[
\lim_{x \to a_i^+} f(x) = f(a_i^+), \quad \lim_{x \to a_{i+1}^-} f(x) = f(a_{i+1}^-)
\]

both exist (and are finite) for each $i = 0, \ldots, N - 1$.

- A function $f(x)$ defined on $\mathbb{R}$ is pwc if it is pwc on every interval $[a, b] \subset \mathbb{R}$.
Pointwise convergence

- Note that, in the theory of Fourier series, if \( f(x) \) is pwc, the value of the function \( f(x) \) at the end points where \( f(x) \) is discontinuous is unimportant (as they do not affect the integral to compute the Fourier coefficients of \( f(x) \)).

- **Definition:** If a function \( f(x) \) defined on \( \mathbb{R} \) is \( 2p \)-periodic \((f(x + 2p) = f(x))\), its Fourier series is the Fourier series of its restriction to the interval \([-p, p]\).

- **Theorem:** Let \( f(x) \) be a \( 2p \)-periodic function defined on \( \mathbb{R} \) such that both \( f(x) \) and \( f'(x) \) are pwc on \( \mathbb{R} \). Then, the Fourier series of \( f(x) \) converges for all \( x \) to a function \( S(x) \) where

\[
S(x) = \begin{cases} 
  f(x), & \text{if } f(x) \text{ is continuous at } x, \\
  \frac{f(x^+) + f(x^-)}{2}, & \text{if } f(x) \text{ is not continuous at } x.
\end{cases}
\]
Section 12.3: Fourier cosine and sine series

• **Definition:** Let $f(x)$ be a function defined on $[-p, p]$
  - $f(x)$ is **even** if $f(-x) = f(x)$.
  - $f(x)$ is **odd** if $f(-x) = -f(x)$.

• Note that if $f(x)$ is even, then $\int_{-p}^{p} f(x) \, dx = 2 \int_{0}^{p} f(x) \, dx$.

• On the other hand, if $f(x)$ is odd, $\int_{-p}^{p} f(x) \, dx = 0$.

• Note that
  - $f(x)$ even and $g(x)$ even $\Rightarrow$ $f(x)g(x)$ even
  - $f(x)$ even and $g(x)$ odd $\Rightarrow$ $f(x)g(x)$ odd
  - $f(x)$ odd and $g(x)$ odd $\Rightarrow$ $f(x)g(x)$ even
Fourier cosine and sine series

• If $f(x)$ is even on $[-p, p]$, we have

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \left( \frac{n\pi x}{p} \right) \, dx = \frac{2}{p} \int_{0}^{p} f(x) \cos \left( \frac{n\pi x}{p} \right) \, dx$$

for $n \geq 0$, and

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \left( \frac{n\pi x}{p} \right) \, dx = 0, \quad n \geq 1.$$  

• Similarly, if $f(x)$ is odd on $[-p, p]$, we have

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \left( \frac{n\pi x}{p} \right) \, dx = 0, \quad n \geq 0$$

and, for $n \geq 1$,

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \left( \frac{n\pi x}{p} \right) \, dx = \frac{2}{p} \int_{0}^{p} f(x) \sin \left( \frac{n\pi x}{p} \right) \, dx$$
Fourier cosine and sine series contd.

- The Fourier series of an even function \( f(x) \) on \([-p, p]\) is thus a Fourier cosine series

\[
f(x) \simeq a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{p} \right) \quad x \in [-p, p],
\]

where

\[a_n = \frac{2}{p} \int_{0}^{p} f(x) \cos \left( \frac{n\pi x}{p} \right) \, dx, \quad n \geq 0\]

- Similarly, the Fourier series of an odd function \( f(x) \) on \([-p, p]\) is a Fourier sine series

\[
f(x) \simeq \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{p} \right) \quad x \in [-p, p],
\]

where

\[b_n = \frac{2}{p} \int_{0}^{p} f(x) \sin \left( \frac{n\pi x}{p} \right) \, dx, \quad n \geq 1\]
Fourier sine series: an example

- The function \( f(x) = \sin(x/2), -\pi < x < \pi \), is odd.
- Its Fourier series on \([-\pi, \pi]\) is thus a sine Fourier series.
- It is given explicitly by

\[
f(x) \simeq \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - 1/4} \frac{n}{\sin(nx)}, \quad x \in [-\pi, \pi].
\]
Half-range expansions; even $2p$-periodic extension

- Suppose that $f(x)$ is defined on the interval $[0, p]$. Then, $f(x)$ can be expanded in a Fourier series in several ways.

- We can, for example, consider the even extension, $f_e(x)$, of $f(x)$ on $[-p, p]$, defined by $f_e(x) = f_e(-x) = f(x)$, $0 < x < p$, and compute its $2p$-periodic cosine Fourier series expansion. The coefficients can be computed directly in terms of the original function $f(x)$.

- We have $f_e(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{p} \right)$ for $x \in [-p, p]$, where, for $n \geq 0$,

$$a_n = \frac{2}{p} \int_{0}^{p} f(x) \cos \left( \frac{n\pi x}{p} \right) \, dx \left( = \frac{2}{p} \int_{0}^{p} f_e(x) \cos \left( \frac{n\pi x}{p} \right) \, dx \right)$$

In particular, since $f_e(x) = f(x)$ for $0 \leq x \leq p$,

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{p} \right) \quad x \in [0, p].$$
Half-range expansions; odd $2p$-periodic extension

- We can also consider the **odd extension**, $f_o(x)$, of $f(x)$ on $[-p, p]$, defined by

$$f_o(x) = \begin{cases} f(x), & 0 < x < p, \\ -f(-x), & -p < x < 0, \end{cases}$$

and compute its $2p$-periodic sine Fourier series expansion. The coefficients can be computed directly in terms of the original function $f(x)$.

- We have $f_o(x) \simeq \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{p} \right)$ for $x \in [-p, p]$, where,

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \left( \frac{n\pi x}{p} \right) \, dx \left( = \frac{2}{p} \int_0^p f_o(x) \sin \left( \frac{n\pi x}{p} \right) \, dx \right),$$

for $n \geq 1$. In particular, since $f_o(x) = f(x)$ for $0 \leq x \leq p$,

$$f(x) \simeq \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{p} \right) \quad x \in [0, p].$$
Half-range expansions; full $p$-periodic Fourier series extension

- A third possibility is to extend $f(x)$ as a $p$-periodic function on the real line ($f(x + p) = f(x)$). The resulting function will have a full Fourier series expansion.

- It is calculated in the same way as for a function defined on $[-p, p]$ except that, in the formulas, $p$ is replaced by $p/2$ and the integration is done over the interval $[0, p]$ instead of $[-p, p]$:

\[
f(x) \simeq a_0 \frac{2}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2n\pi x}{p} \right) + b_n \sin \left( \frac{2n\pi x}{p} \right) \quad x \in [0, p].
\]

where

\[
a_n = \frac{2}{p} \int_{0}^{p} f(x) \cos \left( \frac{2n\pi x}{p} \right) \, dx, \quad n \geq 0,
\]

\[
b_n = \frac{2}{p} \int_{0}^{p} f(x) \sin \left( \frac{2n\pi x}{p} \right) \, dx, \quad n \geq 1.
\]
Section 12.4: Complex Fourier series

- Recall Euler's formula: \( e^{ix} = \cos x + i \sin x \) (and also \( e^{-ix} = \cos x - i \sin x \)).

- If \( f(x) \) is a function defined on \([-p, p]\) its Fourier series

  \[
  f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{p} \right) + b_n \sin \left( \frac{n\pi x}{p} \right) \quad x \in [-p, p],
  \]

  can also be written as

  \[
  f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left( \frac{e^{\frac{in\pi x}{p}} + e^{-\frac{in\pi x}{p}}}{2} \right) + b_n \left( \frac{e^{\frac{in\pi x}{p}} - e^{-\frac{in\pi x}{p}}}{2i} \right) \\
  = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \frac{a_n - i b_n}{2} \right) e^{\frac{in\pi x}{p}} + \sum_{n=1}^{\infty} \left( \frac{a_n + i b_n}{2} \right) e^{-\frac{in\pi x}{p}} \\
  = c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{p}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{p}},
  \]
Complex Fourier series contd.

where the coefficients \( c_n, -\infty < n < \infty \), are defined by:

- \( c_0 = \frac{a_0}{2} = \frac{1}{2p} \int_{-p}^{p} f(x) \, dx \),

\[
c_n = \frac{a_n - i b_n}{2}
\]

\[
= \frac{1}{2p} \int_{-p}^{p} f(x) \cos \left( \frac{n\pi x}{p} \right) \, dx - i \frac{1}{2p} \int_{-p}^{p} f(x) \sin \left( \frac{n\pi x}{p} \right) \, dx
\]

\[
= \frac{1}{2p} \int_{-p}^{p} f(x) \, e^{-i n \pi x/p} \, dx, \quad n \geq 1,
\]

- \( c_{-n} = \frac{a_n + i b_n}{2} = \frac{1}{2p} \int_{-p}^{p} f(x) \, e^{i n \pi x/p} \, dx, \quad n \geq 1.\)
Complex Fourier series contd.

It follows that any (pwc) function $f(x)$ defined on $[-p, p]$ can be expanded as a complex Fourier series

$$f(x) \simeq \sum_{n \in \mathbb{Z}} c_n e^{\frac{in \pi x}{p}},$$

where

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-\frac{in \pi x}{p}} \, dx, \quad n \in \mathbb{Z}.$$  

The complex Fourier series is more elegant and shorter to write down than the one expressed in term of sines and cosines, but it has the disadvantage that the coefficients $c_n$ might be complex even if $f(x)$ is real valued.