Chapt.12: Orthogonal Functions and Fourier series

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12.1:Orthogonal Functions

- Recall that $\mathbb{R}^n = \{(x_1, \ldots, x_n),\ x_i \in \mathbb{R},\ i = 1, \ldots n\}$.
- If $\mathbf{u} = (u_1 \ldots, u_n)$ and $\mathbf{v} = (v_1 \ldots, v_n)$ belong to $\mathbb{R}^n$, their dot product is the number

$$(\mathbf{u}, \mathbf{v}) = u_1\, v_1 + \cdots + u_n\, v_n = \sum_{i=1}^{n} u_i\, v_i.$$

- The dot product has the following properties:
  - $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
  - $(\alpha\, \mathbf{u}, \mathbf{v}) = \alpha\, (\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \alpha\, \mathbf{v}), \quad \alpha \in \mathbb{R}$
  - $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$
  - $(\mathbf{u}, \mathbf{u}) \geq 0$ and $(\mathbf{u}, \mathbf{u}) = 0$ iff $\mathbf{u} = 0$

Orthogonal collections

- The norm of a vector: $\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_n^2} = (\mathbf{u}, \mathbf{u})^{1/2}$
- Orthogonality of two vectors: $\mathbf{u} \perp \mathbf{v}$ iff $(\mathbf{u}, \mathbf{v}) = 0$.
- Orthogonality of a collection of vectors: $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an orthogonal collection of vectors iff $(\mathbf{u}_i, \mathbf{u}_j) = 0$ if $i \neq j$.
- Orthogonal basis: If $m = n$, the dimension of the space, then an orthogonal collection $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ where $\mathbf{u}_i \neq 0$ for all i, forms an orthogonal basis. In that case, any vector $\mathbf{v} \in \mathbb{R}^n$ can be expanded in terms of the orthogonal basis via the formula

$$\mathbf{v} = \sum_{i=1}^{n} (\mathbf{v}, \mathbf{u}_i) \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|^2}.$$

- Orthonormal basis: orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ with $\|\mathbf{u}_i\| = 1$ for all i.

Orthogonal Functions

- In what follows, we will always assume that the functions considered are piecewise continuous on some interval $[a, b]$.
- Inner product: If f_1, f_2 are two functions defined on $[a, b]$, we define their inner product as

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx$$

- Orthogonality: Two functions f_1, f_2 are orthogonal on $[a, b]$ if $(f_1, f_2) = 0$.
- Example: $f(x) = \sin(3x)$, $g(x) = \cos(3x)$. We have

$$\int_{-\pi}^{\pi} \sin(3x)\, \cos(3x)\, dx = 0$$

since $\sin(3x)\, \cos(3x)$ is odd and the interval $[-\pi, \pi]$ is symmetric about 0. Thus $f(x) = \sin(3x)$ and $g(x) = \cos(3x)$ are orthogonal on $[-\pi, \pi]$.

Orthogonal Functions contd.

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- Orthogonal collections: A collection of functions $\{\phi_0(x), \phi_1(x), \ldots, \phi_m(x), \ldots\}$ defined on $[a, b]$ is called orthogonal on $[a, b]$ if

$$(\phi_i, \phi_j) = \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0, \quad \text{when } i \neq j.$$

An example

The collection $\{1, \cos(x), \cos(2x), \cos(3x), \dots\} = \{\cos(k\,x), k \geq 0\}$ is orthogonal on $[-\pi, \pi]$.

- To show this, we use the identity

$$\cos A\, \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}.$$

- We have, if $m, n \geq 0$ are integers with $m \neq n$,

$$\begin{aligned}
&\int_{-\pi}^{\pi} \cos(mx)\, \cos(nx)\, dx \\
&= \int_{-\pi}^{\pi} \frac{\cos((m+n)x) + \cos((m-n)x)}{2}\, dx \\
&= \frac{1}{2}\left[\frac{\sin((m+n)x)}{m+n} + \frac{\sin((m-n)x)}{m-n}\right]_{-\pi}^{\pi} = 0
\end{aligned}$$

Orthonormality

- If $f(x)$ is a function defined on $[a, b]$, we define the norm of f to be

$$\|f\| = (f, f)^{1/2} = \left(\int_a^b f(x)^2 \, dx \right)^{1/2}$$

- A collection of functions $\{\phi_0(x), \phi_1(x), \ldots, \phi_m(x), \ldots\}$ defined on $[a, b]$ is called orthonormal on $[a, b]$ if

$$(\phi_i, \phi_j) = \int_a^b \phi_i(x) \, \phi_j(x) \, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

- Note that if the collection $\{\phi_0(x), \phi_1(x), \ldots, \phi_m(x), \ldots\}$ is orthogonal on $[a, b]$ and $\|\phi_i\| \neq 0$, the collection $\{\frac{\phi_0(x)}{\|\phi_0\|}, \frac{\phi_1(x)}{\|\phi_1\|}, \ldots, \frac{\phi_m(x)}{\|\phi_m\|}, \ldots\}$ is orthonormal on $[a, b]$.

An example

- Consider the collection $\{1, \cos(x), \cos(2x), \cos(3x), \ldots\}$ or $\{\cos(k\,x), k \geq 0\}$ which we have shown to be is orthogonal on $[-\pi, \pi]$.
- We have $\|1\|^2 = \int_{-\pi}^{\pi} 1^2\,dx = 2\,\pi$, so $\|1\| = \sqrt{2\pi}$.
- For $m \geq 1$, we have

$$\|\cos(mx)\|^2 = \int_{-\pi}^{\pi} \cos^2(mx)\,dx = \int_{-\pi}^{\pi} \frac{1+\cos(2mx)}{2}\,dx$$
$$= \left[\frac{x}{2} + \frac{\sin(2mx)}{4m}\right]_{-\pi}^{\pi} = \pi.$$

- Thus $\|\cos(mx)\| = \sqrt{\pi}$.
- The collection $\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\cos(3x)}{\sqrt{\pi}}, \ldots\}$ is thus orthonormal on $[-\pi, \pi]$.

Section 12.1 continued

- Suppose that the collection $\{\phi_n(x)\}_{n\geq 0}$ is an orthogonal collection (or “system”) on $[a, b]$ and that the function $f(x)$ defined on $[a, b]$ can be expanded as a series

$$f(x) = c_0\,\phi_0(x) + c_1\,\phi_1(x) + \cdots + c_n\,\phi_n(x) + \dots, \qquad (1)$$

how can we compute the coefficients $c_0, c_1, c_2, \dots$?

- Note that if (1) holds, we have, for each $n \geq 0$,

$$(f, \phi_n) = \int_a^b f(x)\,\phi_n(x)\,dx = \int_a^b \left\{ \sum_{k=0}^{\infty} c_k\,\phi_k(x) \right\} \phi_n(x)\,dx$$

$$= \sum_{k=0}^{\infty} c_k \int_a^b \phi_k(x)\,\phi_n(x)\,dx = c_n \int_a^b \phi_n^2(x)\,dx = c_n\,\|\phi_n\|^2.$$

Orthogonal systems

- It follows thus that, if (1) holds, then

$$\boxed{c_n = \frac{(f, \phi_n)}{\|\phi_n\|^2}, \quad n \geq 0}\,.$$

and

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n\|^2}\, \phi_n(x). \qquad (2)$$

- However, the expansion formula (2) does not hold in general for an arbitrary orthogonal system on $[a, b]$. For example, it could happen that $f \neq 0$ but $f(x)$ is orthogonal to each function $\phi_n(x)$ in the system and thus the RHS of (2) would be 0 in that case while $f(x) \neq 0$.
- In order for (2) to hold for an arbitrary function $f(x)$ defined on $[a, b]$, there must be "enough" functions ϕ_n in our system.

Completeness

- Definition: An orthogonal system $\{\phi_n(x)\}_{n\geq 0}$ on $[a,b]$ is complete if the fact that a function $f(x)$ on $[a,b]$ satisfies $(f,\phi_n)=0$ for all $n\geq 0$ implies that $f\equiv 0$ on $[a,b]$, or, more precisely, that $\|f\|^2=\int_a^b f^2(x)\,dx=0$.
- If $\{\phi_n(x)\}_{n\geq 0}$ on $[a,b]$ is a complete orthogonal system on $[a,b]$, then every (piecewise continuous) function $f(x)$ on $[a,b]$ has the expansion

$$f(x)\simeq\sum_{n=0}^{\infty}\frac{(f,\phi_n)}{\|\phi_n\|^2}\,\phi_n(x). \qquad (3)$$

on $[a,b]$ in the L^2-sense which means that

$$\lim_{N\to\infty}\int_a^b\left|f(x)-\sum_{n=0}^{N}\frac{(f,\phi_n)}{\|\phi_n\|^2}\,\phi_n(x)\right|^2dx=0.$$

Some remarks

- If $\{\phi_n(x)\}_{n\geq 0}$ on $[a, b]$ is a complete orthogonal system on $[a, b]$, the expansion formula (3) holds for every (pwc) function $f(x)$ on $[a, b]$ in the L^2-sense, but not necessarily "pointwise", i.e. for a fixed $x \in [a, b]$ the series on the RHS of (3) might not necessarily converge and, even if it does, it might not converge to $f(x)$.
- The system $\{1, \cos(x), \cos(2x), \cos(3x), \dots\} = \{\cos(k\,x), k \geq 0\}$ is orthogonal on $[-\pi, \pi]$ but it is **not complete** on $[-\pi, \pi]$.
- Indeed, if $f(x)$ any odd function on $[-\pi, \pi]$ ($f(-x) = -f(x)$) with $\|f\| \neq 0$, such as $f(x) = x$ or $f(x) = \sin x$, we have

$$\int_{-\pi}^{\pi} f(x)\,\cos(n\,x)\,dx = 0, \quad n \geq 0,$$

since $f(x)\,\cos(n\,x)$ is odd and $[-\pi, \pi]$ is symmetric about 0.

Section 12.2: Fourier series

- Theorem: The system

$$\mathcal{T} := \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \dots\}$$

is a complete orthogonal system on $[-\pi, \pi]$.

- To show the orthogonality of this system, one needs to show that

$$\int_{-\pi}^{\pi} \cos(mx)\ \cos(nx)\, dx = 0, \quad m, n \geq 0, m \neq n, \ (a)$$

$$\int_{-\pi}^{\pi} \sin(mx)\ \sin(nx)\, dx = 0, \quad m, n \geq 1, m \neq n, \ (b)$$

$$\int_{-\pi}^{\pi} \cos(mx)\ \sin(nx)\, dx = 0, \quad m \geq 0, n \geq 1. \ (c)$$

Fourier series contd.

- For example, to show (b), we use the formula

$$\sin A \, \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}.$$

- We have then, if $m, n \geq 1$ and $m \neq n$,

$$\begin{aligned} &\int_{-\pi}^{\pi} \sin(mx) \, \sin(nx) \, dx \\ &= \int_{-\pi}^{\pi} \frac{\cos((m-n)x) - \cos((m+n)x)}{2} \, dx \\ &= \left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n} \right]_{-\pi}^{\pi} = 0. \end{aligned}$$

- We have also, for $m, n \geq 1$,

$$\|1\|^2 = 2\pi, \quad \|\cos(mx)\|^2 = \pi, \quad \|\sin(nx)\|^2 = \pi.$$

Fourier series expansions

- Note that the completeness of the system $\mathcal{T}$ is much more difficult to prove.
- Using the previous theorem, it follows that every (pwc) function $f(x)$ on $[-\pi, \pi]$ admits the expansion

$$\boxed{f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\}} \quad (4),$$

where $\frac{a_0}{2} = \frac{(f,1)}{\|1\|^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx =$ average of f on $[-\pi, \pi]$,

$$a_n = \frac{(f, \cos(nx))}{\|\cos(nx)\|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \;\; n \geq 1,$$

$$b_n = \frac{(f, \sin(nx))}{\|\sin(nx)\|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \;\; n \geq 1.$$

Fourier series on general intervals

- The series expansion (4) in terms of the trigonometric system $\mathcal{T}$ is called the Fourier series expansion of $f(x)$ on $[-\pi, \pi]$.
- More generally, if $p > 0$ and $f(x)$ is pwc on $[-p, p]$, then it will have a Fourier series expansion on $[-p, p]$ given by

$$\boxed{f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right\}} \quad (4),$$

where the Fourier coefficients a_n, b_n are defined by

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right) dx, \ n \geq 0,$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx \ n \geq 1.$$

An example

- The function

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 < x < \pi \end{cases}$$

has a Fourier series expansion on $[-\pi, \pi]$ given by

$$f(x) \simeq \frac{\pi}{4} + \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left(\frac{-2}{\pi n^2} \right) \cos(nx) + \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sin(nx)$$

$$\simeq \frac{\pi}{4} - \frac{2}{\pi} \cos(x) - \frac{2}{9\pi} \cos(3x) - \frac{2}{25\pi} \cos(5x) + \dots$$

$$\sin(x) - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) - \frac{1}{4} \sin(4x) + \dots \qquad (*)$$

Periodic extension

- If a function $f(x)$ defined on the interval $[-p,p]$ is expanded as the Fourier series

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right\} \quad (5),$$

  we can view the RHS of (5) as a function defined om all of $\mathbb{R}$.

- Since

$$\cos\left(\frac{n\pi(x+2p)}{p}\right) = \cos\left(\frac{n\pi x}{p} + 2n\pi\right) = \cos\left(\frac{n\pi x}{p}\right),$$

$$\sin\left(\frac{n\pi(x+2p)}{p}\right) = \sin\left(\frac{n\pi x}{p} + 2n\pi\right) = \sin\left(\frac{n\pi x}{p}\right),$$

  the RHS of (5) is $2p$-periodic and thus equal to the $2p$-periodic extension of $f(x)$ to the real line.

Piecewise continuity

- Recall that a function $f(x)$ defined on the interval $[a, b]$ is piecewise continuous (pwc) on $[a, b]$ if $[a, b]$ can be divided into N subintervals $[a_i, a_{i+1}]$, $i = 0, \ldots, N-1$ with $a = a_0 < a_1 < a_2 < \cdots < a_{N-1} < a_N = b$ and such that $f(x)$ is continuous on each open interval (a_i, a_{i+1}), $i = 0, \ldots, N-1$ and

$$\lim_{x \to a_i^+} f(x) = f(a_i^+), \qquad \lim_{x \to a_{i+1}^-} f(x) = f(a_{i+1}^-)$$

  both exist (and are finite) for each $i = 0, \ldots, N-1$.

- A function $f(x)$ defined on $\mathbb{R}$ is pwc if it is pwc on every interval $[a, b] \subset \mathbb{R}$.

Pointwise convergence

- Note that, in the theory of Fourier series, if $f(x)$ is pwc, the value of the function $f(x)$ at the end points a_i where $f(x)$ is discontinuous is unimportant (as they do not affect the integral to compute the Fourier coefficients of $f(x)$).
- Definition: If a function $f(x)$ defined on $\mathbb{R}$ is $2p$-periodic ($f(x+2p) = f(x)$), its Fourier series is the Fourier series of its restriction to the interval $[-p, p]$.
- Theorem: Let $f(x)$ be a $2p$-periodic function defined on $\mathbb{R}$ such that both $f(x)$ and $f'(x)$ are pwc on $\mathbb{R}$. Then, the Fourier series of $f(x)$ converges for all x to a function $S(x)$ where

$$\boxed{S(x) = \begin{cases} f(x), & \text{if } f(x) \text{ is continuous at } x, \\ \frac{f(x^+)+f(x^-)}{2}, & \text{if } f(x) \text{ is not continuous at } x. \end{cases}}$$

Section 12.3: Fourier cosine and sine series

- Definition: Let $f(x)$ be a function defined on $[-p, p]$
  - $f(x)$ is even if $f(-x) = f(x)$.
  - $f(x)$ is odd if $f(-x) = -f(x)$.
- Note that if $f(x)$ is even, then $\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx$.
- On the other hand, if $f(x)$ is odd, $\int_{-p}^{p} f(x)\,dx = 0$.
- Note that
  - $f(x)$ even and $g(x)$ even $\Longrightarrow f(x)\,g(x)$ even
  - $f(x)$ even and $g(x)$ odd $\Longrightarrow f(x)\,g(x)$ odd
  - $f(x)$ odd and $g(x)$ odd $\Longrightarrow f(x)\,g(x)$ even

Fourier cosine and sine series

- If $f(x)$ is even on $[-p,p]$, we have

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx$$

  for $n \geq 0$, and

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\left(\frac{n\pi x}{p}\right)\,dx = 0,\ \ n \geq 1.$$

- Similarly, if $f(x)$ is odd on $[-p,p]$, we have

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx = 0,\ \ n \geq 0$$

  and, for $n \geq 1$,

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\left(\frac{n\pi x}{p}\right)dx = \frac{2}{p}\int_0^p f(x)\,\sin\left(\frac{n\pi x}{p}\right)dx$$

Fourier cosine and sine series contd.

- The Fourier series of an even function $f(x)$ on $[-p,p]$ is thus a Fourier cosine series

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) \quad x \in [-p,p], \qquad (6)$$

where $\boxed{a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx, \ n \geq 0}$.

- Similarly, the Fourier series of an odd function $f(x)$ on $[-p,p]$ is a Fourier sine series

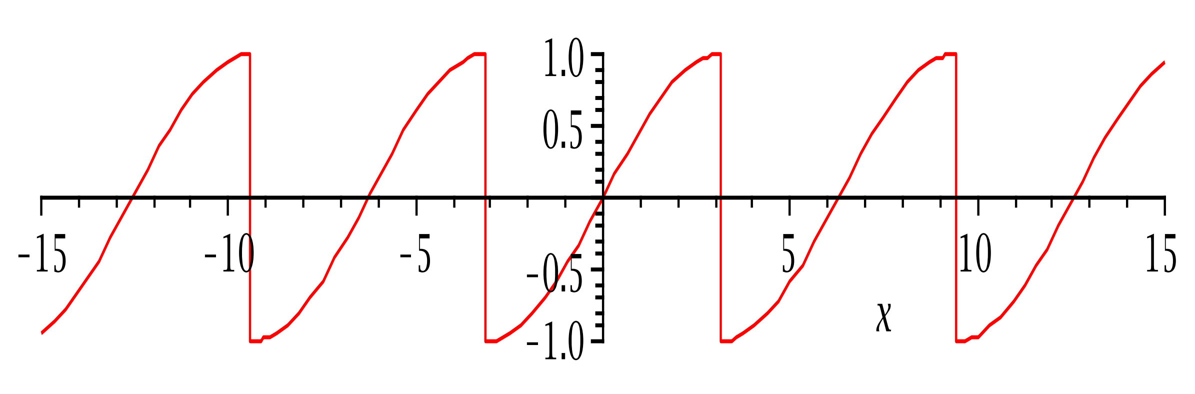
$$f(x) \simeq \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right) \quad x \in [-p,p], \qquad (7)$$

where $\boxed{b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx, \ n \geq 1}$.

Fourier sine series:an example

- The function $f(x) = \sin(x/2)$, $-\pi < x < \pi$, is odd.
- Its Fourier series on $[-\pi, \pi]$ is thus a sine Fourier series.
- It is given explicitly by

$$f(x) \simeq \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 - 1/4} \sin(nx), \quad x \in [-\pi, \pi].$$



Half-range expansions; even $2p$-periodic extension

- Suppose that $f(x)$ is defined on the interval $[0, p]$. Then, $f(x)$ can be expanded in a Fourier series in several ways.
- We can, for example, consider the even extension, $f_e(x)$, of $f(x)$ on $[-p, p]$, defined by $f_e(x) = f_e(-x) = f(x),\ 0 < x < p$, and compute its $2p$-periodic cosine Fourier series expansion. The coefficients can be computed directly in terms of the original function $f(x)$.
- We have $f_e(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$ for $x \in [-p, p]$, where, for $n \geq 0$,

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx \left(= \frac{2}{p}\int_0^p f_e(x)\cos\left(\frac{n\pi x}{p}\right)dx\right)$$

In particular, since $f_e(x) = f(x)$ for $0 \leq x \leq p$,

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) \quad x \in [0, p].$$

Half-range expansions; odd $2p$-periodic extension

- We can also consider the odd extension, $f_o(x)$, of $f(x)$ on $[-p, p]$, defined by

$$f_o(x) = \begin{cases} f(x), & 0 < x < p, \\ -f(-x), & -p < x < 0, \end{cases}$$

and compute its $2p$-periodic sine Fourier series expansion. The coefficients can be computed directly in terms of the original function $f(x)$.

- We have $f_o(x) \simeq \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$ for $x \in [-p, p]$, where,

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx \left(= \frac{2}{p}\int_0^p f_o(x)\sin\left(\frac{n\pi x}{p}\right)dx\right),$$

for $n \geq 1$. In particular, since $f_o(x) = f(x)$ for $0 \leq x \leq p$,

$$f(x) \simeq \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right) \qquad x \in [0, p].$$

Half-range expansions; full p-periodic Fourier series extension

- A third possibility is to extend $f(x)$ as a p-periodic function on the real line ($f(x+p) = f(x)$). The resulting function will have a full Fourier series expansion.
- It is calculated in the same way as for a function defined on $[-p,p]$ except that, in the formulas, p is replaced by $p/2$ and the integration is done over the interval $[0,p]$ instead of $[-p,p]$:
-

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{p}\right) + b_n \sin\left(\frac{2n\pi x}{p}\right) \qquad x \in [0,p].$$

where

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{2n\pi x}{p}\right)dx, \;\; n \geq 0,$$

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{2n\pi x}{p}\right)dx, \;\; n \geq 1.$$

Section 12.4: Complex Fourier series

- Recall Euler's formula: $e^{ix} = \cos x + i \sin x$ (and also $e^{-ix} = \cos x - i \sin x$).
- If $f(x)$ is a function defined on $[-p, p]$ its Fourier series

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \qquad x \in [-p, p],$$

can also be written as

$$\begin{aligned} f(x) &\simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{\frac{in\pi x}{p}} + e^{-\frac{in\pi x}{p}}}{2}\right) + b_n \left(\frac{e^{\frac{in\pi x}{p}} - e^{-\frac{in\pi x}{p}}}{2i}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - i\, b_n}{2}\right) e^{\frac{in\pi x}{p}} + \sum_{n=1}^{\infty} \left(\frac{a_n + i\, b_n}{2}\right) e^{-\frac{in\pi x}{p}} \\ &= c_0 + \sum_{n=1}^{\infty} c_n\, e^{\frac{in\pi x}{p}} + \sum_{n=1}^{\infty} c_{-n}\, e^{-\frac{in\pi x}{p}}, \end{aligned}$$

Complex Fourier series contd.

where the coefficients c_n, $-\infty < n < \infty$, are defined by:

- $c_0 = \frac{a_0}{2} = \frac{1}{2\,p} \int_{-p}^{p} f(x)\,dx$,
- $$\begin{aligned} c_n &= \frac{a_n - i\,b_n}{2} \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx - i\,\frac{1}{2p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)dx \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)\,e^{\frac{-i\,n\,\pi\,x}{p}}\,dx, \quad n \geq 1, \end{aligned}$$
- $$c_{-n} = \frac{a_n + i\,b_n}{2} = \frac{1}{2p}\int_{-p}^{p} f(x)\,e^{\frac{i\,n\,\pi\,x}{p}}\,dx, \quad n \geq 1.$$

Complex Fourier series contd.

It follows that any (pwc) function $f(x)$ defined on $[-p, p]$ can be expanded as a complex Fourier series

- $$\boxed{f(x) \simeq \sum_{n\in\mathbb{Z}} c_n \, e^{\frac{in\pi x}{p}}} \,,$$

  where

- $$\boxed{c_n = \frac{1}{2p} \int_{-p}^{p} f(x) \, e^{\frac{-in\pi x}{p}} \, dx, \quad n \in \mathbb{Z}} \,.$$

- The complex Fourier series is more elegant and shorter to write down than the one expressed in term of sines and cosines, but it has the disadvantage that the coefficients c_n might be complex even if $f(x)$ is real valued.