Applications of triple integrals

If $\rho(x,y,z)$ denotes the mass density (per unit of volume) at (x,y,z) in some solid region V, then the total mass of V is

$$\boxed{m = \iiint_V \rho(x,y,z)\, dV}$$

We also define the 1st moments with respect to the coordinate planes:

$$\boxed{M_{yz} = \iiint_V x\,\rho(x,y,z)\,dV, \quad M_{xz} = \iiint_V y\,\rho(x,y,z)\,dV, \quad M_{xy} = \iiint_V z\,\rho(x,y,z)\,dV.}$$

The center of mass (or centroid if $\rho(x,y,z)=1$) is the point $(\bar{x},\bar{y},\bar{z})$ where

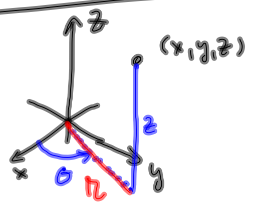
$$\boxed{\bar{x} = \frac{M_{yz}}{m} = \frac{\iiint_V x\,\rho(x,y,z)\,dV}{\iiint_V \rho(x,y,z)\,dV}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}}$$

Moments of inertia about the coordinate axes:

$$M_x = \iiint_V (y^2+z^2)\,\rho(x,y,z)\,dV, \quad M_y = \iiint_V (x^2+z^2)\,\rho(x,y,z)\,dV$$

$$M_z = \iiint_V (x^2+y^2)\,\rho(x,y,z)\,dV$$

Cylindrical coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

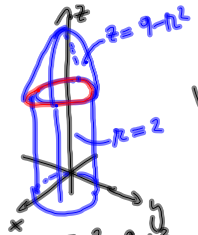
If V is a solid region in the x,y,z space and V^* is that same region expressed in cylindrical coordinates, then

$$\boxed{\iiint_V f(x,y,z)\,dV = \iiint_{V^*} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz}$$

Ex: Let V be the solid region above the x,y plane bounded by the cylinder $x^2+y^2=4$ and below the paraboloid $z = 9-x^2-y^2$.

Compute $\iiint_V \sqrt{x^2+y^2}\, z\, dV$

Sol. The cylinder $r^2=4$ or $r=2$ and the paraboloid $z = 9-r^2$ intersect when $r=2$, $z=5$



$$V^* = \{(r,\theta,z),\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 2-r^2\}$$

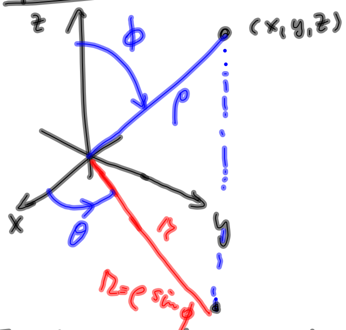
$$\iiint_V \sqrt{x^2+y^2}\, z \, dV$$

$$= \int_0^{2\pi}\int_0^2\int_0^{9-r^2} (r z)\, r \, dz\, dr\, d\theta = 2\pi \int_0^2 \int_0^{9-r^2} r^2 z \, dz\, dr$$

$$= 2\pi \int_0^2 r^2 \left[\frac{z^2}{2}\right]_{z=0}^{z=9-r^2} dr = 2\pi \int_0^2 r^2 \frac{(9-r^2)^2}{2}\, dr$$

$$= \frac{4{,}168}{35}\pi .$$

Spherical coordinates



$$\begin{cases} x = \rho \cos\theta \sin\phi \\ y = \rho \sin\theta \sin\phi \\ z = \rho \cos\phi \end{cases}$$

$0 \le \phi \le \pi$

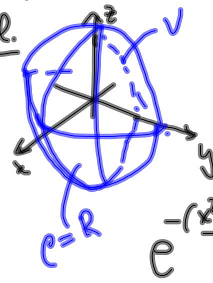
$0 \le \theta \le 2\pi$

If V is a solid region in the x,y,z space and V^* is that same region expressed in spherical coordinates, then we have

$$\boxed{\iiint_V f(x,y,z)\, dV = \iiint_{V^*} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}$$

Ex: A star occupies a spherical region V centered at $(0,0,0)$ with radius R. The mass density (per unit of volume) at (x,y,z) is given by $g(x,y,z) = e^{-\left(\frac{x^2+y^2+z^2}{a^2}\right)^{3/2}}$, $a>0$. Find the total mass m of the star.

Sol.



$$m = \iiint_V g(x,y,z)\, dV$$

$$V^* = \{(\rho,\theta,\phi),\ 0 \le \rho \le R,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

Since $x^2+y^2+z^2 = \rho^2$,

$$e^{-\left(\frac{x^2+y^2+z^2}{a^2}\right)^{3/2}} = e^{-(\rho^2/a^2)^{3/2}} = e^{-(\rho^3/a^3)}$$

$$\therefore m = \iiint_V e^{-\left(\frac{x^2+y^2+z^2}{a^2}\right)^{3/2}} dV = \int_0^{2\pi}\int_0^{\pi}\int_0^R e^{-\rho^3/a^3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \underbrace{\left[-\cos\phi\right]_0^{\pi}}_{2} \left[-\frac{e^{-\rho^3/a^3} a^3}{3}\right]_0^R = \frac{4\pi}{3} a^3 \left(1 - e^{-R^3/a^3}\right)$$

$\square$