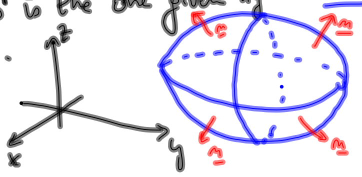
9.16 The divergence theorem.

Positive orientation.

If V is a closed, bounded solid in $\mathbb{R}^3$ enclosed by a closed surface S. Then, the positive orientation on S is the one given by the outward pointing normal.



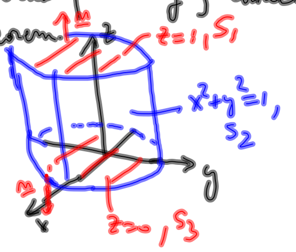
The divergence theorem.

Let V be a solid region enclosed by a closed surface S which is positively oriented (as above). If $\underline{F}(x,y,z)$ is a vector field with continuous 1st order partial derivatives on a neighborhood of V, then

$$\boxed{\iint_S \underline{F}\cdot d\underline{S} = \iiint_V \nabla\cdot\underline{F}\, dV}$$

Ex: Compute $\iint_S \underline{F}\cdot d\underline{S}$ where $\underline{F}(x,y,z) = \underline{i}+\underline{j}+z(x^2+y^2)\underline{k}$ and S is the cylinder $x^2+y^2=1$, $0\le z\le 1$ together with the disks $\begin{matrix}x^2+y^2\le 1\\ z=0\end{matrix}$, $\begin{matrix}x^2+y^2\le 1\\ z=1\end{matrix}$, oriented positively, directly and using the divergence theorem.

Sol.



S consists of 3 surfaces:
S_1 = "the top"
S_2 = the part on the cylinder
S_3 = "the bottom"

On S_1: $\underline{r}(x,y) = \langle x, y, 1\rangle$ for $x^2+y^2\le 1$

$\underline{r}_x = \langle 1,0,0\rangle$, $\underline{r}_x\times\underline{r}_y = \langle 0,0,1\rangle$

$\underline{r}_y = \langle 0,1,0\rangle$, Let $D = \{(x,y) \mid x^2+y^2\le 1\}$

$$\iint_{S_1}\underline{F}\cdot d\underline{S} = \iint_D \langle 1,1,x^2+y^2\rangle\cdot\langle 0,0,1\rangle\, dA$$

$$= \iint_D x^2+y^2\, dx\,dy \underset{\text{polar coord.}}{=} \int_0^{2\pi}\int_0^1 r^2\, r\, dr\, d\theta = 2\pi\left[\frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}.$$

On S_2: $\underline{r}(\theta,z) = \langle\cos\theta, \sin\theta, z\rangle$, $0\le\theta\le 2\pi$, $0\le z\le 1$.

$\underline{r}_\theta = \langle -\sin\theta, \cos\theta, 0\rangle$, $\underline{r}_\theta\times\underline{r}_z =$

$\underline{r}_z = \langle 0, 0, 1\rangle$, $\langle\cos\theta, \sin\theta, 0\rangle$ ↑ correct orientation

$$\iint_{S_2}\underline{F}\cdot d\underline{S} = \int_0^{2\pi}\int_0^1 \langle 1,1,z\rangle\cdot\langle\cos\theta,\sin\theta,0\rangle\, dz\, d\theta$$

$$= \int_0^{2\pi}\cos\theta+\sin\theta\, d\theta = \left[\sin\theta-\cos\theta\right]_0^{2\pi} = 0$$

On S_3: The unit normal is $\underline{n} = \langle 0,0,-1\rangle$
and $\underline{F} = \langle 1,1,0\rangle$

$$\therefore \ \underline{F}\cdot\underline{n} = 0$$

$$\therefore \iint_{S_3} \underline{F}\cdot d\underline{S} = \iint_{S_3} (\underbrace{\underline{F}\cdot\underline{n}}_{0})\, dS = 0$$

$$\therefore \iint_{S} \underline{F}\cdot d\underline{S} = \frac{\pi}{2} + 0 + 0 = \frac{\pi}{2}.$$

Using the divergence theorem:

$$\nabla\cdot\underline{F} = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}\left(z(x^2+y^2)\right) = x^2+y^2.$$

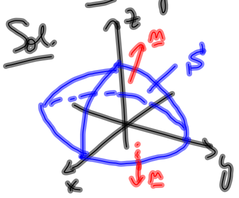
$$\iiint_V \nabla\cdot\underline{F}\, dV = \iiint_V x^2+y^2\, dV \underset{\text{Cylindrical Coord.}}{=} \int_0^{2\pi}\int_0^1\int_0^1 r^2\, r\, dr\, dz\, d\theta$$

$$= 2\pi\left[\frac{r^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{4}\right) = \frac{\pi}{2}.$$

Ex: Let V be the solid region under the sphere $x^2+y^2+z^2=1$ and above the x,y plane. If the boundary of V, S, is positively oriented, compute the flux of the vector field

$$\underline{F} = yz\,\underline{i} + xz\,\underline{j} + xy\,\underline{k} \ \text{ across } S$$

Sol.



$$\nabla\cdot\underline{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$$

$$\therefore \iint_S \underline{F}\cdot d\underline{S} = \iiint_V \underbrace{\nabla\cdot\underline{F}}_{0}\, dV = 0 \qquad \square$$

Ex: Same problem with $\underline{F} = x^3\,\underline{i} + y^3\,\underline{j} + z^3\,\underline{k}$.

Sol. $$\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2+y^2+z^2)$$

$$\therefore \iint_S \underline{F}\cdot d\underline{S} = \iiint_V \nabla\cdot\underline{F}\, dV = \iiint_V 3(x^2+y^2+z^2)\, dV$$

Passing to spherical coord., $V^* = \{(\rho,\theta,\phi),\ 0\le\rho\le 1,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{2}\}$

$$\iiint_V 3(x^2+y^2+z^2)\, dV = 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$$

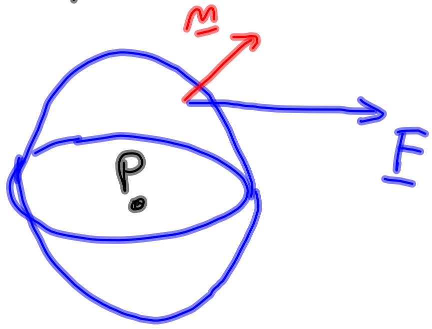
$$3(2\pi)\int_0^{\pi/2}\sin\phi\, d\phi \int_0^1 \rho^4\, d\rho$$

$$= 6\pi\left[-\cos\phi\right]_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_0^1$$

$$= 6\pi\,(1)\left(\frac{1}{5}\right) = \frac{6\pi}{5} \qquad \square$$

Interpretation of the divergence

Let $\underline{F}(x,y,z)$ represents the velocity field of a fluid (at some time t_0). Let $\rho(x,y,z)$ be the mass density of the fluid at (x,y,z). Let S be a small sphere centered at P. The outward flow across S (if S is positively oriented) is given by:



$$\iint_S \rho\, \underline{F} \cdot \underline{n}\, dS$$

By the divergence theorem,

$$\iint_S \rho\, \underline{F} \cdot \underline{n}\, dS = \iint_S (\rho \underline{F}) \cdot \underline{dS} = \iiint_V \nabla \cdot (\rho \underline{F})\, dV$$

(where V is the solid region inside the sphere)

If the radius is small and $\operatorname{div}(\rho \underline{F})$ is continuous,

$$\iiint_V \nabla \cdot (\rho \underline{F})\, dV \approx \nabla \cdot (\rho \underline{F})(P) \times (\operatorname{Vol} V)$$

$\therefore\ \nabla \cdot (\rho \underline{F})(P) =$ outward flow per unit of volume at P.

If $\nabla \cdot (\rho \underline{F})(P) > 0$, P is called a <u>source</u>

< 0 " " <u>sink</u>