Continuity equation.

Let F represents the velocity field of a fluid at time t which now varies and let $\rho(x,y,z,t)$ be the mass density of the fluid at (x,y,z) and time t. Let P be a point in the fluid and let S be a small sphere centred at P. The total mass of the solid region V enclosed by S at time t is

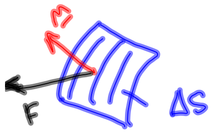
$$m = \iiint_V \rho(x,y,z,t)\, dx\, dy\, dz$$

$\therefore$ The rate of change of that mass with respect to time is thus

$$\frac{dm}{dt} = \iiint_V \frac{\partial \rho}{\partial t}(x,y,z,t)\, dt.$$

The volume of fluid flowing through an element of surface ΔS per unit of time is $\approx \underline{F}.n \ \text{Area}(\Delta S)$

$$= \underline{F}.\underline{n}\ ds$$ area of ΔS



The mass of fluid flowing through ΔS per unit of time is $\approx \rho\ \underline{F}.\underline{n}\ ds$

$\therefore$ The total mass of fluid flowing out of S per unit of time is then

$$\iint_S \rho\, \underline{F}.\underline{n}\, ds = \iiint_V \nabla.(\rho \underline{F})\, dV$$

divergence theorem

$$= \frac{dm}{dt} = \iiint_V \frac{\partial \rho}{\partial t}(x,y,z,t)\, dV$$

Since the point P and the radius of the sphere are arbitrary, we conclude that

$$\boxed{\frac{\partial \rho}{\partial t} = \operatorname{div}(\rho \underline{F})}$$

Continuity equation

In particular, if $\rho = \text{Constant}$ (fluid is "incompressible")

then $\frac{\partial \rho}{\partial t} = 0 = \operatorname{div}(\rho \underline{F}) \Longrightarrow$

$$\boxed{\operatorname{div}(\underline{F}) = 0}$$

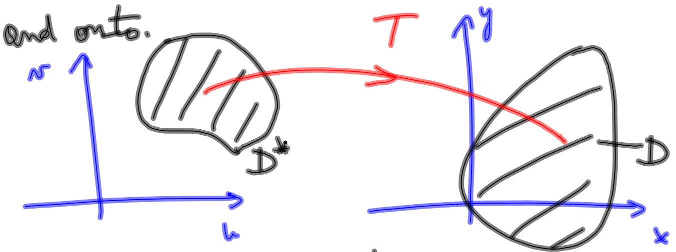
9.17 Change of Variables in multiple integrals

Two dimensional Case

Suppose that the rectangular Coordinates x, y are related to another Coordinate system u, v by:

$$\boxed{x = g_1(u,v), \quad y = g_2(u,v)} \quad (1)$$

We denote by T the transformation from the u,v plane to the x,y plane defined by (1). We will assume that T is of class C^1 (which means that $g_1, g_2, \frac{\partial g_1}{\partial x}, \frac{\partial g_1}{\partial y}, \frac{\partial g_2}{\partial x}, \frac{\partial g_2}{\partial y}$ are continuous on the domain in which they are defined) and T maps the region D^* in the u,v plane to the region D in the x,y plane and that $T: D^* \to D$ is one-to-one and onto.



Def The Jacobian of the transformation T at the point (u,v) is defined by

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

Theorem (Change of Variables formula)

If $f(x,y)$ is integrable on D,

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv}$$

Where $T: D^* \to T$ is 1-1, onto and of class C^1.

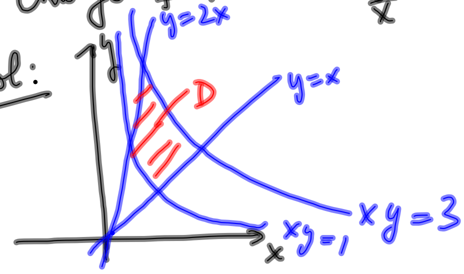
Ex: Polar Coordinates

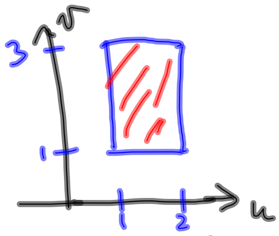
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex: Evaluate $\iint_D x^3 y \, dx\, dy$, where D is the region in the 1^{st} quadrant of the xy plane bounded by the lines $y=x$, $y=2x$ and by the hyperbolas $xy=1$ and $xy=3$, using the change of variables $\frac{y}{x}=u$, $xy=v$.

Sol:





Note that $D=\{(x,y),\ 1\le \underbrace{xy}_{v}\le 3,\ 1\le \underbrace{\frac{y}{x}}_{u}\le 2\}$

so $D^*=\{(u,v),\ 1\le u\le 2,\ 1\le v\le 3\}$

We need to express x,y in terms of u,v

$$uv=\frac{y}{x}\,xy=y^2 \Rightarrow y=\sqrt{uv}=u^{1/2}v^{1/2}$$

$$x=\frac{v}{y}=u^{-1/2}v^{1/2}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}-\frac{1}{2}u^{-3/2}v^{1/2} & \frac{1}{2}u^{-1/2}v^{-1/2}\\ \frac{1}{2}u^{-1/2}v^{\frac{1}{2}} & \frac{1}{2}u^{1/2}v^{-1/2}\end{vmatrix}$$

$$=-\frac{1}{4}u^{-1}-\frac{1}{4}u^{-1}=-\frac{u^{-1}}{2}$$

Note: $\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$ (check!)

$$\therefore \iint_D x^3 y\,dx\,dy=\iint_{D^*} u^{-3/2}v^{3/2}\,u^{1/2}v^{1/2}\left(\frac{u^{-1}}{2}\right)du\,dv$$

$$=\frac{1}{2}\iint_D u^{-2}v^2\,du\,dv=\frac{1}{2}\int_1^2 u^{-2}\,du\int_1^3 v^2\,dv$$

$$=\frac{1}{2}\left[-u^{-1}\right]_1^2\left[\frac{v^3}{3}\right]_1^3$$

$$=\frac{1}{2}\left(1-\frac{1}{2}\right)\left(9-\frac{1}{3}\right)=\frac{1}{4}\,\frac{26}{3}=\frac{13}{6}$$

$\square$