3 dimensional Case.

Let V and V^* be two solid region in $\mathbb{R}^3$ and let $T: V^* \longrightarrow V$ be a mapping of class C^1 which is 1-1 and onto.

Let $T(u,v,w) = \langle x(u,v,w), y(u,v,w), z(u,v,w)\rangle$.

If $f(x,y,z)$ is integrable on V, then

$$\boxed{\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V^*} f(T(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw,}$$

where
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Jacobian

Ex: Cylindrical Coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\boxed{\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V^*} f(r\cos\theta, r\sin\theta, z)\; r\,dr\,d\theta\,dz.}$$

Ex: Spherical Coordinate

$$\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases} \qquad \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi\,(-\rho^2\sin\phi\cos\phi) - \rho\sin\phi\,(\rho\sin^2\phi)$$

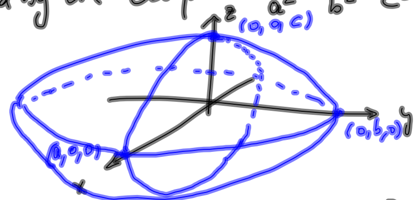
$$= -\rho^2\sin\phi\,[\underbrace{\cos^2\phi + \sin^2\phi}_{1}] = -\rho^2\sin\phi$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi$$

$$\boxed{\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V^*} f(T(\rho,\theta,\phi))\;\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

Ex: Compute the Volume of the Solid region V enclosed by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. $(a,b,c>0)$

Sol.



$$V=\left\{(x,y,z),\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}.$$

let $\frac{x}{a}=u$, $\frac{y}{b}=v$, $\frac{z}{c}=w$ or $x=ua$, $y=vb$, $z=wc$

$$\therefore T(u,v,w)=(au,bv,cw)$$

then $V^*=\{(u,v,w),\ u^2+v^2+w^2\le 1\}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}a&0&0\\0&b&0\\0&0&c\end{vmatrix}=abc$$

$$\therefore Vol(V)=\iiint_V 1\,dx\,dy\,dz=\iiint_{V^*} 1\ |abc|\ du\,dv\,dw$$

$$=abc\ Vol(V^*)\underset{\text{spherical coord.}}{=}abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=abc\ 2\pi\int_0^{\pi}\sin\phi\,d\phi\int_0^1\rho^2\,d\rho$$

$$=abc\ 2\pi\left[-\cos\phi\right]_0^{\pi}\left[\frac{\rho^3}{3}\right]_0^1=abc\ 2\pi\,(2)\left(\frac{1}{3}\right)$$

$$=\frac{4\pi}{3}abc.$$

$\square$

13.1 : Reading assignment.

Chapter 13 : Partial differential Eq.

Important tools: Fourier Sine and Cosine Series

Sine Fourier Series : If $f(x)$ is pwc on $[0,L]$, we can expand it as the Sine Fourier Series

$$f(x)\simeq\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right),\quad 0<x<L,$$

where $b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$, $n\ge 1$

Cosine Fourier Series: If $f(x)$ is pwc on $[0,L]$, we can also expand it as the Cosine Fourier Series

$$f(x)\simeq\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right),\quad 0<x<L$$

where $a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $n\ge 0$

Eigenvalue problems:

Ex: Find the non-trivial solutions (i.e. non identically zero) of

$$y''(x) + \lambda\, y(x) = 0, \quad 0 < x < L$$

satisfying $y(0) = y(L) = 0$

Sol. The auxiliary eq. is $m^2 + \lambda = 0$

3 cases: $\lambda < 0, \lambda = 0, \lambda > 0$

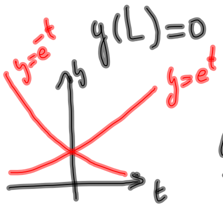
Case 1: $\lambda < 0$ $\quad \lambda = -\omega^2 \ (\omega > 0)$

$m^2 - \omega^2 = 0$ so $m = \pm \omega$

gen. sol. is $y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow y(x) = c_1(e^{\omega x} - e^{-\omega x})$
$\quad c_2 = -c_1$

$y(L) = 0 \Rightarrow c_1 \underbrace{(e^{\omega L} - e^{-\omega L})}_{\neq 0} = 0 \Rightarrow c_1 = 0$



no non-trivial solution

Case 2: $\lambda = 0$ $\quad m^2 = 0$

$y(x) = c_1 + c_2 x$

$y(0) = 0 \Rightarrow c_1 = 0$, so $y = c_2 x$

$y(L) = 0 \Rightarrow c_2 \underbrace{L}_{\neq 0} = 0 \Rightarrow c_2 = 0$

No non-trivial sol.

Case 3: $\lambda > 0$: $\lambda = \omega^2 \ (\omega > 0)$

aux. eq. $m^2 + \omega^2 = 0$, i.e. $m = \pm i\omega$

$\therefore\ y(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$

$y(0) = 0 \Rightarrow c_1 = 0$, so $y(x) = c_2 \sin(\omega x)$

$y(L) = 0 \Rightarrow c_2 \sin(\omega L) = 0$

For non-trivial solutions to exist, we need:

$\sin(\omega L) = 0$, i.e. $\omega L = k\pi$, k integer, $k \geq 1$

$\therefore$ For each $k \geq 1$, we get the eigenvalue

$\lambda_k = \left(\frac{k\pi}{L}\right)^2$ and a corresponding

eigenfunction: $\phi_k(x) = \sin\left(\frac{k\pi x}{L}\right)$, $k \geq 1$

Rem: In a similar way, one can show that the eigenvalues and eigenfunctions for the eigenvalue problem

$$\begin{cases} y''(x) + \lambda\, y(x) = 0, \quad 0 < x < L \\ y'(0) = y'(L) = 0 \end{cases}$$

are $\lambda_k = \left(\frac{k\pi x}{L}\right)^2$, $k \geq 0$

$\phi_0(x) = 1$, $\quad \phi_k(x) = \cos\left(\frac{k\pi x}{L}\right)$, $k \geq 1$