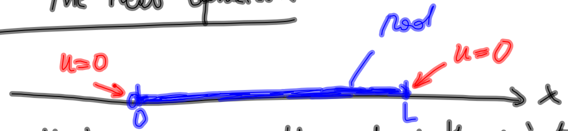
B.3 The heat equation



$u(x,t)$ is the temperature on the rod at the point x ($0 \le x \le L$) and time $t \ge 0$.

It can be shown that $u(x,t)$ satisfies: ($k>0$)

$$\begin{cases} (1) & \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}, \; 0<x<L,\; t>0 & \text{(heat equation)} \\ (2) & u(0,t) = u(L,t) = 0,\; t>0 & \text{(boundary conditions)} \\ (3) & u(x,0) = f(x),\; 0<x<L & \text{(initial condition)} \end{cases}$$

where $f(x)$ is the initial temperature distribution on the rod. To solve the heat equation, we first look for particular solutions of (1) and (2) of the form

$$\boxed{u(x,t) = X(x)\,T(t)} \quad (4)$$

("Separation of Variables")

Using (1) and (4), we get

$$\frac{\partial u}{\partial t} = X(x)T'(t) = k\frac{\partial^2 u}{\partial x^2} = k\,X''(x)T(t)$$

Dividing by $kX(x)T(t)$, we get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}, \quad \begin{matrix} t>0 \\ 0<x<L \end{matrix}$$

both terms must be equal to the same constant!

$\therefore$ there exists a number λ such that

$$\boxed{\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda}$$

The condition (2) shows that $X(0)T(t) = X(L)T(t) = 0$ for any $t>0$. For the solution to be non-trivial (i.e. non identically zero), we need

$$\boxed{X(0) = X(L) = 0}$$

We first try to find solutions for $X(x)$ of

Eigenvalue problem
$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}$$

Non-trivial solutions only exist for

$$\lambda = \lambda_m = \left(\frac{m\pi}{L}\right)^2, \quad m = 1,2,3,4,\ldots$$

and the corresponding solution is $X_m(x) = \sin\left(\frac{m\pi x}{L}\right)$.

For each $m \ge 1$, we also need to solve

$$\frac{T'(t)}{kT(t)} = -\left(\frac{m\pi}{L}\right)^2 \quad \text{or} \quad T'(t) = -\left(\frac{m\pi}{L}\right)^2 kT(t)$$

The solution of this 1st order DE is

$$T_m(t) = A_m e^{-\left(\frac{m\pi}{L}\right)^2 kt}$$

$\therefore$ We found a family of solutions of (1) and (2) of the form:

$$\boxed{u_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}}$$

<u>Superposition principle</u>

If $u_1(x,t), u_2(x,t), u_3(x,t), \dots$ are solutions of (1) and (2), then so is the sum of the series

$$\sum_{n=1}^{\infty} C_n u_n(x,t) \quad \text{(if the series converges)}$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

is solution of (1) and (2).

We now choose the coefficients A_n so that (3) holds:

$$u(x,0) = f(x) = \underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)}_{\text{sine Fourier series expansion of } f(x) \text{ on } (0,L)}, \quad 0<x<L$$

$$\therefore \quad A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1 \qquad (*)$$

The solution is thus given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt},$$

where A_n is given by $(*)$.

<u>Rem:</u> If the boundary points are insulated, the boundary conditions (2) change to

$$\boxed{\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \quad (2')}$$

A similar computation yields:

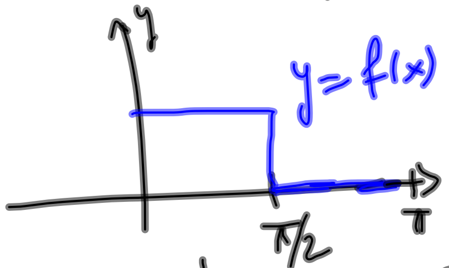
$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

where $a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0$

Ex: Solve the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = (0.1)\dfrac{\partial^2 u}{\partial x^2}, & 0<x<\pi \\ & t>0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

where $f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$



Sol. $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/2}\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi/2} = \frac{2}{\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$\therefore$ the solution is thus

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin(nx)\, e^{-n^2(0.1)t}$$

Ex: Same problem with

$$f(x) = \sin(2x) + \frac{1}{5}\sin(7x)$$

Sol. By inspection:

$$b_2 = 1, \quad b_7 = \frac{1}{5}, \quad b_n = 0 \text{ otherwise}$$

$$u(x,t) = \sin(2x)\, e^{-4(0.1)t} + \frac{1}{5}\sin(7x)\, e^{-4.9t}$$

$\square$