Case 3: $\lambda < 0$ $\quad \lambda = -\omega^2 \ (\omega > 0)$

The aux. eq. is $m^2 + \omega^2 = 0$ and has roots $m = \pm i\omega$

the general solution is $y(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x)$

$$y'(x) = \omega\left[-C_1 \sin(\omega x) + C_2 \cos(\omega x)\right]$$

$$y'(0) = 0 \Rightarrow \underbrace{\omega}_{>0} C_2 = 0 \Rightarrow C_2 = 0$$

$$y'(L) = 0 \Rightarrow \omega\left[-C_1 \sin(\omega L)\right] = 0$$

Non-trivial solutions exist only if $\sin(\omega L) = 0$

i.e. $\omega L = k\pi$, with $k = 1, 2, 3, \ldots$

$\therefore$ We get non-trivial solutions of the form

$$\boxed{\phi_k(x) = C_k \cos\left(\frac{k\pi x}{L}\right), \quad k = 1, 2, 3, \ldots}$$

Adding the solution found in (b), we obtain all the non-trivial solutions ("eigenfunctions") as:

$$\boxed{\phi_k(x) = C_k \cos\left(\frac{k\pi x}{L}\right), \quad k = 0, 1, 2, 3, \ldots}$$

## Sturm-Liouville problem.

Let $q(x), r(x), r'(x), p(x)$ be continuous functions on $[a,b]$ with $r(x) > 0$, $p(x) > 0$ for $x \in [a,b]$.

Consider the eigenvalue problem:

$$\begin{cases} (r(x)y'(x))' + q(x)y(x) + \lambda p(x) y(x) = 0 \\ A_1 y(a) + B_1 y'(a) = 0 \\ A_2 y(b) + B_2 y'(b) = 0 \end{cases} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right) \text{Boundary conditions}$$

Then, there exist an infinite number of real eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$$

$$\text{with } \lim_{n\to\infty} \lambda_n = \infty$$

and associated eigenfunctions $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$

Furthermore, the collection $\{\phi_n(x)\}_{n \geq 1}$ is a complete orthogonal system on $[a,b]$.

Ex: $(r(x) = 1,\ q(x) = 0,\ p(x) = 1)$

(a) $\begin{cases} y'' + \lambda y = 0, \quad 0 < x < L \\ y(0) = y(L) = 0 \end{cases}$ eigenvalues: $\left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$

eigenfunctions: $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

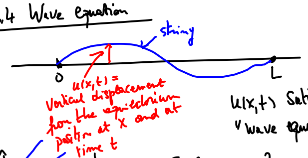
$n = 1, 2, 3, \ldots$

(b) $\begin{cases} y'' + \lambda y = 0, \quad 0 < x < L \\ y'(0) = y'(L) = 0 \end{cases}$ eigenvalues: $\left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, 3, \ldots$

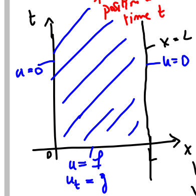
$\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

$n = 0, 1, 2, 3, \ldots$

## 13.4 Wave equation




u(x,t) satisfies the "wave equation"




$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 & (1) \\ u(0,t) = u(L,t) = 0, \quad t > 0 & (2) \\ u(x,0) = f(x),\ u_t(x,0) = g(x), \quad 0 < x < L & (3) \end{cases}$$

We will the method of "Separation of Variables" to find non-trivial solutions of (1) and (2).

i.e. we look first for solutions of the form

$$\boxed{u(x,t) = X(x)\,T(t)}$$

$u(x,t) = X(x)\,T(t)$, $\quad u_{tt} = a^2 u_{xx}$

From (1), we get $\quad X(x)T''(t) = a^2 X''(x)\,T(t)$, or
(Dividing by $a^2 T X$)

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)}, \quad 0<x<L$$

$$\therefore \frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = C, \quad 0<x<L$$

From (2), $\quad X(0)T(t) = 0 = X(L)T(t)$, for all $t>0$

$$\Longrightarrow X(0) = X(L) = 0$$

We first solve the eq. for $X(x)$: $\begin{cases} X''(x) - \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}$

(eigenvalue problem)

Non-trivial solutions only exist when $C = -\left(\frac{m\pi}{L}\right)^2$ where $m = 1,2,3,\ldots$ and are of the form $\boxed{X_m(x) = \sin\left(\frac{m\pi x}{L}\right)}$

For each $m\geq 1$, we need to solve

$$T''(t) = -\left(\frac{m\pi}{L}\right)^2 a^2\, T(t) \text{ or } T''(t) + \left(\frac{m\pi}{L}a\right)^2 T(t) = 0$$

the aux. eq. is $m^2 + \left(\frac{m\pi a}{L}\right)^2 = 0$ has solutions $m = \pm i\frac{m\pi a}{L}$

$\therefore$ For each $m\geq 1$, we get the corresponding solution

$$\boxed{T_m(t) = A_m \cos\left(\frac{m\pi a t}{L}\right) + B_m \sin\left(\frac{m\pi a t}{L}\right)}$$

$\therefore$ For each $m\geq 1$, we get particular solutions of (1) and (2) of the form

$$u_m(x,t) = \left[A_m \cos\left(\frac{m\pi a t}{L}\right) + B_m \sin\left(\frac{m\pi a t}{L}\right)\right] \times \sin\left(\frac{m\pi x}{L}\right)$$

By the superposition principle, the series $\sum_{m=1}^{\infty} u_m(x,t)$ is also a solution of (1), (2) if it converges

$$\therefore u(x,t) = \sum_{m=1}^{\infty} \left[A_m \cos\left(\frac{m\pi a t}{L}\right) + B_m \sin\left(\frac{m\pi a t}{L}\right)\right] \sin\left(\frac{m\pi x}{L}\right)$$

To solve (3), we need

$$u(x,0) = f(x) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{L}\right), \quad 0<x<L.$$

$\therefore$ the $A_m$ have to be sine Fourier coefficients of $g$ on $(0,L)$: $\Longrightarrow \quad A_m = \frac{2}{L}\int_0^l f(x) \sin\left(\frac{m\pi x}{L}\right) dx$, $m\geq 1$

$$u_t(x,0) = \sum_{m=1}^{\infty} \frac{m\pi a}{L}\left[-A_m \underbrace{\sin\left(\frac{m\pi a}{L}0\right)}_{0} + B_m \underbrace{\cos\left(\frac{m\pi a}{L}0\right)}_{1}\right] \sin\left(\frac{m\pi x}{L}\right)$$

$$= \sum_{m=1}^{\infty} \frac{m\pi a}{L} B_m \sin\left(\frac{m\pi x}{L}\right) = g(x), \quad 0<x<L$$

$\Rightarrow \frac{m\pi a}{L} B_m =$ sine Fourier coeff. of $g$ on $(0,L)$ $= \frac{2}{L}\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx$

$$\therefore \boxed{\begin{aligned} u(x,t) &= \sum_{m=1}^{\infty} \left[A_m \cos\left(\frac{m\pi a}{L}t\right) + B_m \sin\left(\frac{m\pi a}{L}t\right)\right] \sin\left(\frac{m\pi x}{L}\right) \\ \text{where } A_m &= \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx \\ B_m &= \frac{2}{m\pi a}\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx, \quad m\geq 1 \end{aligned}}$$