Wave equation:
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, & 0<x<L,\ t>0 \quad (1) \\ u(0,t) = u(L,t), & t>0 \quad (2) \\ u(x,0) = f(x), & u_t(x,0) = g(x) \quad (3) \end{cases}$$

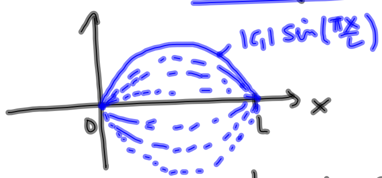
The solution is given by:
$$u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right) \right\} \sin\left(\frac{n\pi x}{L}\right),$$
where
$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$
$$B_n = \frac{2}{n\pi a}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Note that $A_n \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right)$

can be written as $C_n \sin\left(\frac{n\pi a t}{L} + \phi_n\right)$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi a t}{L} + \phi_n\right) \sin\left(\frac{n\pi x}{L}\right)$$

In particular, $u_1(x,t) = C_1 \sin\left(\frac{\pi a t}{L} + \phi_1\right) \sin\left(\frac{\pi x}{L}\right)$
is the 1st <u>standing wave</u>.



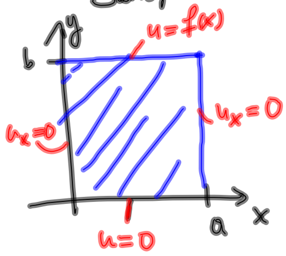
$$u_2(x,t) = C_2 \sin\left(\frac{2\pi a t}{L} + \phi_2\right) \sin\left(\frac{2\pi x}{L}\right)$$
"2d standing wave"

<u>Ex:</u> Solve the wave equation
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & 0<x<\pi,\ t>0 \\ u(0,t) = u(\pi,t) = 0, & t>0 \\ u(x,0) = \sin x, & u_t(x,0) = \sin(2x),\ 0<x<\pi \end{cases}$$

<u>Sol.</u> The solution has the form:
$$u(x,t) = \sum_{n=1}^{\infty} \{A_n \cos(nt) + B_n \sin(nt)\} \sin(nx).$$
$$u(x,0) = f(x) = \sin x = \sum_{n=1}^{\infty} A_n \sin(nx), \quad 0<x<\pi$$
$\therefore$ $A_1 = 1$ and $A_n = 0$, $n \neq 1$.
$$u_t(x,0) = g(x) = \sin(2x) = \sum_{n=1}^{\infty} n B_n \sin(nx)$$
$\therefore$ $B_2 = \frac{1}{2}$ and $B_n = 0$, $n \neq 2$.
$$\therefore \quad u(x,t) = \cos t \sin x + \frac{1}{2}\sin(2t)\sin(2x)$$
$\square$

B.5 Laplace equation

The steady-state temperature in a rectangular plate whose vertical edges are insulated satisfies:



$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0, \quad 0<x<a,\ 0<y<b & (1) \\ \dfrac{\partial u}{\partial x}(0,y)=\dfrac{\partial u}{\partial x}(a,y)=0, \quad 0<y<b & (2) \\ u(x,0)=0, \quad 0<x<a & (3) \\ u(x,b)=f(x), \quad 0<x<a & (4) \end{cases}$$

Separation of variables: look for non-trivial solutions of (1), (2), (3) of the form $\boxed{u(x,y)=X(x)\,Y(y)}$

(1) $\Rightarrow$ $X''(x)Y(y)+X(x)Y''(y)=0$

or $\dfrac{X''(x)}{X(x)}+\dfrac{Y''(y)}{Y(y)}=0$, or $\dfrac{X''(x)}{X(x)}=-\dfrac{Y''(y)}{Y(y)}=-\lambda$

(2) $\Rightarrow$ $X'(0)=X'(a)=0$.

$\therefore$ $X(x)$ is a non-trivial solution of the eigenvalue problem $\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=X'(a)=0 \end{cases}$

$\therefore$ $\lambda=\lambda_m=\left(\dfrac{m\pi}{a}\right)^2$, $m=0,1,2,3,4,\dots$

and the corresponding eigenfunction are

$$\boxed{X_m(x)=C_m\cos\left(\frac{m\pi x}{a}\right)}$$

For each $m\geq 0$, we need to solve:

$$Y''(y)-\left(\frac{m\pi}{a}\right)^2 Y(y)=0$$

(3) $\Rightarrow$ $Y(0)=0$

$\underline{m=0}$ $\begin{cases} Y''(y)=0 \\ Y(0)=0 \end{cases}$ $\Rightarrow$ $Y(y)=B_0+A_0 y$

$Y(0)=0 \Rightarrow B_0=0$

$\therefore$ $\boxed{Y_0(y)=A_0 y}$

$\underline{m\geq 1}$ The aux. eq. is $m^2-\left(\frac{m\pi}{a}\right)^2=0$

or $m=\pm\left(\frac{m\pi}{a}\right)$

The gen. sol. is $Y(y)=A_m' e^{\frac{m\pi}{a}y}+B_m' e^{-\frac{m\pi}{a}y}$

$Y(0)=0 \Rightarrow A_m'+B_m'=0$

$\Rightarrow$ $Y_m(y)=A_m'\left(e^{\frac{m\pi}{a}y}-e^{-\frac{m\pi}{a}y}\right)$

$=A_m\sinh\left(\frac{m\pi y}{a}\right)$ (where $A_m=2A_m'$)

$\therefore$ For each $m\geq 0$, we get particular solution $u_m(x,y)$ of the form $u_m(x,y)=X_m(x)\,Y_m(y)$

with $u_0(x,y)=C_0\, y$

$u_m(x,y)=C_m\cos\left(\frac{m\pi x}{a}\right)\sinh\left(\frac{m\pi y}{a}\right)$

By the superposition principle, we get

$$u(x,y) = C_0 y + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

is a solution of (1), (2), (3).

To solve (4), we need

$$u(x,b) = f(x) = \underbrace{C_0 b + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right)}_{\text{Cosine Fourier series}}$$

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right), \ 0<x<a$

to be continued ....