13.5 Laplace equation, continued

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0<x<a,\ 0<y<b & (1) \\ \dfrac{\partial u}{\partial x}(0,y) = \dfrac{\partial u}{\partial x}(a,y) = 0, \quad 0<y<b & (2) \\ u(x,0) = 0, \quad 0<x<a & (3) \\ u(x,b) = f(x), \quad 0<x<a & (4) \end{cases}$$

The general solution of (1), (2), (3) has the form

$$\boxed{u(x,y) = C_0 y + \sum_{m=1}^{\infty} C_m \cos\left(\frac{m\pi x}{a}\right) \sinh\left(\frac{m\pi y}{a}\right)}$$

To solve (4), we need

$$u(x,b) = f(x) = C_0 b + \sum_{m=1}^{\infty} C_m \sinh\left(\frac{m\pi b}{a}\right) \cos\left(\frac{m\pi x}{a}\right), \quad 0<x<a$$

Cosine Fourier series expansion of $f(x)$ on $(0,a)$

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{a}\right),$$

where $a_m = \dfrac{2}{a} \displaystyle\int_0^a f(x) \cos\left(\frac{m\pi x}{a}\right) dx$

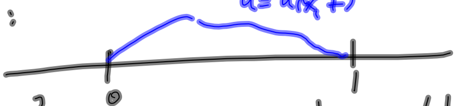
$$\therefore \quad C_0 b = \frac{a_0}{2} \Rightarrow \boxed{C_0 = \frac{a_0}{2b} = \frac{1}{ab} \int_0^a f(x)\,dx}$$

If $m \geq 1$, $C_m \sinh\left(\frac{m\pi b}{a}\right) = a_m$

$$\therefore \quad \boxed{C_m = \frac{2}{a \sinh\left(\frac{m\pi b}{a}\right)} \int_0^a f(x) \cos\left(\frac{m\pi x}{a}\right) dx, \quad m \geq 1.}$$

13.6 Non-homogeneous PDEs

How to solve the basic "Non-homogeneous" heat equation:



$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} + r, \quad 0<x<1,\ t>0, \quad (k>0) & (1) \\ u(0,t) = 0, \quad u(1,t) = u_0, \quad t>0 & (2) \\ u(x,0) = f(x), \quad 0<x<1 & (3) \end{cases}$$

This models a situation where heat is generated on the rod at a constant rate of r.

Let $\boxed{u(x,t) = v(x,t) + \psi(x)}$

Idea: chose $v(x,t)$ and $\psi(x)$ so that $v(x,t)$ is solution of the corresponding homogeneous problem (i.e. as above with $r=0$, $u_0=0$)

We have $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \psi''(x)$, $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}$

$\therefore \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + r \Rightarrow \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + k\psi''(x) + r$

Since we want $v(x,t)$ to satisfy $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$,

we need $\psi(x)$ to satisfy $k\psi''(x) + r = 0$, $0<x<1$

$\therefore \psi''(x) = -\frac{r}{k}$, so $\psi'(x) = -\frac{r x}{k} + c_1$

and $\psi(x) = -\frac{r x^2}{2k} + c_1 x + c_2$

Using the boundary conditions:

$u(0,t) = 0 = v(0,t) + \psi(0)$

$u(1,t) = u_0 = v(1,t) + \psi(1)$.

Since we want $v(x,t)$ to satisfy $v(0,t) = v(1,t) = 0$,

we need $\psi(0) = 0$, $\psi(1) = u_0$.

$\psi(0) = 0 \Rightarrow c_2 = 0$, so $\psi(x) = -\frac{r x^2}{2k} + c_1 x$

$\psi(1) = u_0 \Rightarrow -\frac{r}{2k} + c_1 = u_0 \Rightarrow c_1 = u_0 + \frac{r}{2k}$

$\therefore \psi(x) = -\frac{r x^2}{2k} + \left(\frac{r}{2k} + u_0\right)x$

and $v(x,t)$ is determined by solving

$$(*)\quad \begin{cases} \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}, & 0<x<1,\ t>0 \\ v(0,t) = v(1,t) = 0, & t>0 \\ v(x,0) = u(x,0) - \psi(x) \\ \qquad = f(x) - \psi(x) \\ \qquad = f(x) + \frac{r x^2}{2k} - \left(\frac{r}{2k} + u_0\right)x \end{cases}$$

We know that the solution of this homogeneous heat equation is given by:

$$v(x,t) = \sum_{m=1}^{\infty} A_m e^{-k(m\pi)^2 t} \sin(m\pi x),\quad 0<x<1,$$

where $A_m = \frac{2}{1}\int_0^1 \left\{ f(x) + \frac{r x^2}{2k} - \left(\frac{r}{2k} + u_0\right)x \right\} \sin(m\pi x)\,dx$, $m \geq 1$

and the solution of the original problem (i.e. the non-homogeneous heat equation) is thus given by:

$$u(x,t) = \underbrace{-\frac{r}{2k}x^2 + \left[u_0 + \frac{r}{2k}\right]x}_{\text{steady-state solution}} + \underbrace{\sum_{m=1}^{\infty} A_m e^{-k(m\pi)^2 t} \sin(m\pi x)}_{\to 0 \text{, exponentially as } t \to \infty},\quad 0<x<1,\ t>0$$

transient term