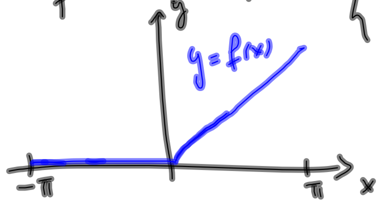
Ex: Compute the Fourier Series of the function $f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 < x \leq \pi \end{cases}$



Sol.: $(p = \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$$

If $m \geq 1$,

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos(mx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\frac{\sin(mx)}{m}\right]' dx = \frac{1}{\pi}\left\{\underbrace{\left[\frac{x\sin(mx)}{m}\right]_0^{\pi}}_{0} - \int_0^{\pi} 1\,\frac{\sin(mx)}{m}\,dx\right\}$$

$$= \frac{1}{\pi}\left[\frac{\cos(mx)}{m^2}\right]_0^{\pi} = \frac{1}{\pi m^2}\left[\cos(m\pi) - 1\right] = \frac{(-1)^m - 1}{\pi m^2}$$

$$= \begin{cases} 0, & \text{if } m \text{ is even} \\ \frac{-2}{\pi m^2}, & \text{if } m \text{ is odd} \end{cases}$$

If $m \geq 1$, $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin(mx)\,dx$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[-\frac{\cos(mx)}{m}\right]' dx$$

$$= \frac{1}{\pi}\left\{\left[-\frac{x\cos(mx)}{m}\right]_0^{\pi} + \int_0^{\pi} 1\,\frac{\cos(mx)}{m}\,dx\right.$$

$$= \frac{1}{\pi}\left\{\frac{-\pi\cos(m\pi)}{m} + \underbrace{\left[\frac{\sin(mx)}{m^2}\right]_0^{\pi}}_{0}\right\}$$

$$= -\frac{(-1)^m}{m} = \frac{(-1)^{m+1}}{m}.$$

$$\therefore\ f(x) \approx \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos(mx) + b_m\sin(mx)$$

or

$$f(x) \approx \frac{\pi}{4} + \sum_{\substack{m=1 \\ m\ \text{odd}}}^{\infty}\left(\frac{-2}{\pi m^2}\right)\cos(mx) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin(mx)$$

□