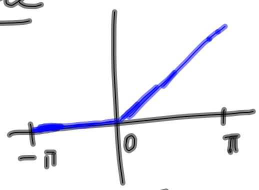
Connection to previous example

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 < x < \pi \end{cases}$$



If $m \geq 1$,

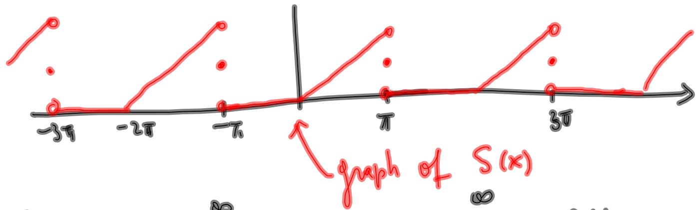
$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos(mx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\frac{\sin(mx)}{m}\right]' dx = \frac{1}{\pi}\left\{\underbrace{\left[\frac{x\sin(mx)}{m}\right]_0^{\pi}}_{0} - \int_0^{\pi} \frac{\sin(mx)}{m}\,dx\right\}$$

$$= \frac{1}{\pi}\left[\frac{\cos(mx)}{m^2}\right]_0^{\pi} = \frac{1}{\pi m^2}\left[\cos(m\pi) - 1\right]$$

$$= \frac{1}{\pi m^2}\left[(-1)^m - 1\right] = \begin{cases} 0, & \text{if } m \text{ is even} \\ \dfrac{-2}{\pi m^2}, & \text{if } m \text{ is odd} \end{cases}$$

Ex: $f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 < x < \pi \end{cases}$



$$f(x) \simeq \frac{\pi}{4} + \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty}\left(\frac{-2}{\pi m^2}\right)\cos(mx) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin(mx)$$

Since $f(x)$ is continuous at $x=0$ and $f(0)=0$, then $S(0)=0$

$$\therefore\ 0 = \frac{\pi}{4} + \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty}\frac{-2}{\pi m^2}\underbrace{\cos(0)}_{1} + \sum_{m=1}\frac{(-1)^{m+1}}{m}\underbrace{\sin(0)}_{0}$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right\}$$

$$\therefore\ \boxed{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}}$$

On the other hand, $f(x)$ is discontinuous at $x=\pi$, so $S(\pi) = \dfrac{f(\pi^+) + f(\pi^-)}{2} = \dfrac{0+\pi}{2} = \dfrac{\pi}{2}$

Indeed,

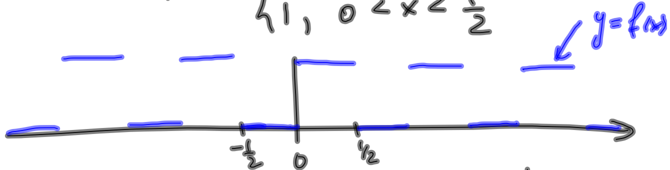
$$S(\pi) = \frac{\pi}{4} + \sum_{\substack{m=1 \\ m\,\text{odd}}}^{\infty} \frac{-2}{\pi m^2} \underbrace{\cos(m\pi)}_{\substack{(-1)^m \\ \| \\ -1 \text{ since } m \text{ is odd}}} + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \underbrace{\sin(m\pi)}_{0}$$

$$\therefore \quad \frac{\pi}{4} + \sum_{\substack{m=1 \\ m\,\text{odd}}}^{\infty} \frac{2}{\pi m^2} = \frac{\pi}{4} + \frac{2}{\pi} \underbrace{\sum_{\substack{m=1 \\ m\,\text{odd}}}^{\infty} \frac{1}{m^2}}_{\frac{\pi^2}{8}}$$

$$= \frac{\pi}{2} = \frac{f(\pi^+) + f(\pi^-)}{2}$$

Ex: • Find the Fourier series expansion of the 1-periodic function $f(x)$ such that

$$f(x) = \begin{cases} 0, & -\frac{1}{2} < x < 0 \\ 1, & 0 < x < \frac{1}{2} \end{cases}$$



• Use the FS expansion obtained to compute the sum of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots$$

Sol: $p = \frac{1}{2}$

$$a_0 = 2\int_{-1/2}^{1/2} f(x)\,dx = 2\int_0^{1/2} 1\,dx = 2\,\frac{1}{2} = 1$$

$$a_m = 2\int_{-1/2}^{1/2} f(x)\cos(2m\pi x)\,dx = 2\int_0^{1/2} \cos(2m\pi x)\,dx$$

$$= 2\left[\frac{\sin(2m\pi x)}{2m\pi}\right]_0^{1/2} = 0, \quad m \geq 1.$$

$$b_m = 2\int_{-1/2}^{1/2} f(x)\sin(2m\pi x)\,dx = 2\int_0^{1/2} \sin(2m\pi x)\,dx$$

$$= 2\left[-\frac{\cos(2m\pi x)}{2m\pi}\right]_0^{1/2} = \frac{1}{m\pi}\left[1 - \cos(m\pi)\right]$$

$$= \begin{cases} 0, & \text{if } m \text{ is even} \\ \frac{2}{m\pi}, & \text{if } m \text{ is odd} \end{cases}$$

$$\therefore \quad f(x) \approx \frac{1}{2} + \sum_{\substack{m=1 \\ m\,\text{odd}}}^{\infty} \frac{2}{m\pi} \sin(2m\pi x)$$

$$= \frac{1}{2} + \frac{2}{\pi}\left\{ \frac{\sin(2\pi x)}{1} + \frac{\sin(6\pi x)}{3} + \frac{\sin(10\pi x)}{5} + \dots \right\}$$