Ex: The function $f(x) = \begin{cases} 0, & -\frac{1}{2} < x < 0 \\ 1, & 0 \le x < \frac{1}{2} \end{cases}$

has the 1-periodic Fourier series expansion

$$f(x) \simeq \frac{1}{2} + \frac{2}{\pi}\left\{\frac{\sin(2\pi x)}{1} + \frac{\sin(6\pi x)}{3} + \frac{\sin(10\pi x)}{5} + \dots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2(1+2k)\pi x)}{1+2k}$$

How to evaluate the sum

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

For $x = \frac{1}{4}$, we get

$$\frac{1}{2} + \frac{2}{\pi}\left\{\sin(\pi/2) + \frac{\sin(\frac{3\pi}{2})}{3} + \frac{\sin(\frac{5\pi}{2})}{5} + \dots\right\}$$

$$= \frac{1}{2} + \frac{2}{\pi}\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right\}$$

$= f\left(\frac{1}{4}\right) = 1$ (Since $f(x)$ is continuous at $x = \frac{1}{4}$)

$$\therefore\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots = \frac{\pi}{4}. \quad \square$$

Ex: Expand the function $\sin(x/2)$ as a sine Fourier series on $[-\pi, \pi]$.

Sol. Note: $\sin(-\frac{x}{2}) = -\sin(\frac{x}{2})$, so $f(x)$ is odd on $[-\pi, \pi]$.

$\therefore\ a_m = 0$, for all $m \geq 0$

Also, $b_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(x/2)\sin(mx)\,dx$$

We use: $\boxed{\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}}$

$$\therefore\ b_m = \frac{1}{\pi}\int_0^{\pi} \cos((m-\tfrac{1}{2})x) - \cos((m+\tfrac{1}{2})x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin((m-\frac{1}{2})x)}{m-\frac{1}{2}} - \frac{\sin((m+\frac{1}{2})x)}{m+\frac{1}{2}}\right]_0^{\pi}$$

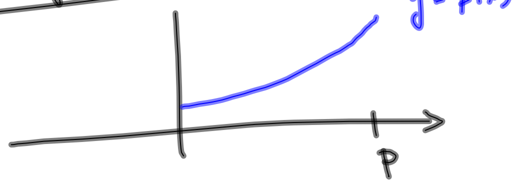
$$= \frac{1}{\pi}\left\{\frac{\sin((m-\frac{1}{2})\pi)}{m-\frac{1}{2}} - \frac{\sin((m+\frac{1}{2})\pi)}{m+\frac{1}{2}}\right\}$$

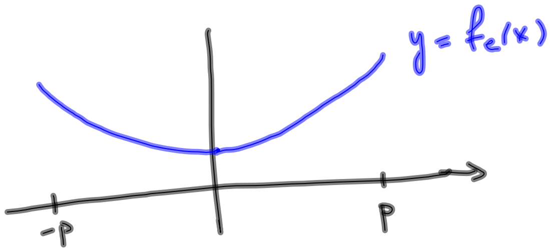
Since $\sin((m-\frac{1}{2})\pi) = (-1)^{m+1}$, $\sin((m+\frac{1}{2})\pi) = (-1)^m$

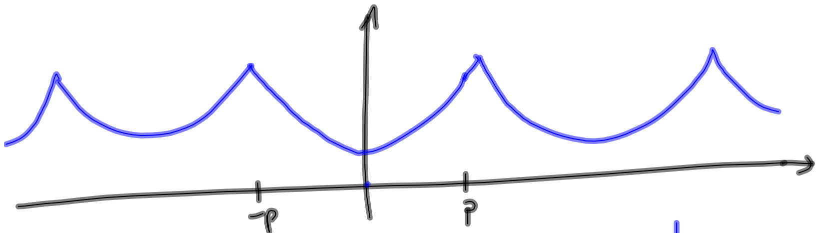
$$\therefore\ b_m = \frac{(-1)^{m+1}}{\pi}\left[\frac{1}{m-1/2} + \frac{1}{m+1/2}\right] = \frac{(-1)^{m+1}}{\pi}\frac{2m}{m^2 - 1/4}$$

$$\therefore\ f(x) \approx \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{\pi}\frac{2m}{m^2 - \frac{1}{4}}\sin(mx) \quad \square$$

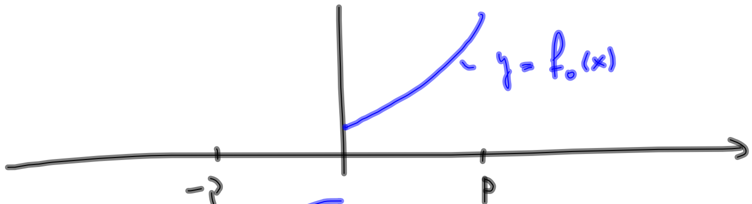
# Half-range expansions

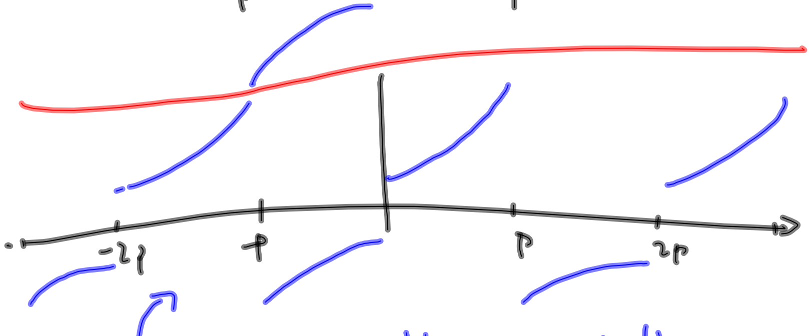


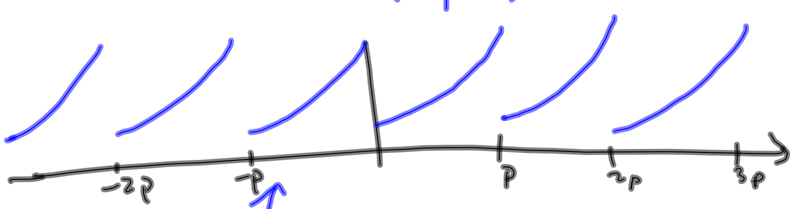




graph of the even 2p-periodic extension of f(x)




graph of the odd 2p-periodic extension of f(x)




graph of the p-periodic extension of f(x)