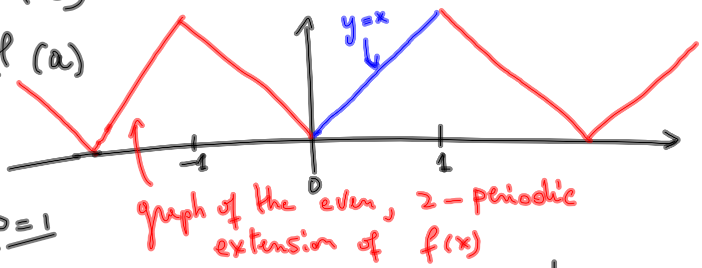
__Ex:__ Expand $f(x) = x$, $0<x<1$

(a) as a cosine Fourier series (of period 2)

(b) as a sine Fourier series (of period 2)

(c) as a Fourier series (of period 1)

__Sol__ (a)



$\underline{P=1}$

$$a_m = \frac{2}{1}\int_0^1 f(x)\cos\left(\frac{m\pi x}{1}\right)dx = 2\int_0^1 x\cos(m\pi x)dx$$

If $m=0$, $a_0 = 2\int_0^1 x\, dx = 2\left[\frac{x^2}{2}\right]_0^1 = 1$

If $m\geq 1$, $a_m = 2\int_0^1 x\cos(m\pi x)dx = 2\int_0^1 x\left(\frac{\sin(m\pi x)}{m\pi}\right)'dx$

$$= \left\{\underbrace{\left[\frac{2x\sin(m\pi x)}{m\pi}\right]_0^1}_{0} - 2\int_0^1 1\cdot\frac{\sin(m\pi x)}{m\pi}dx\right\}$$

$$= \left[\frac{2\cos(m\pi x)}{(m\pi)^2}\right]_0^1 = \frac{2}{(m\pi)^2}\left(\cos(m\pi)-1\right)$$

$$= \frac{2}{(m\pi)^2}\left((-1)^m - 1\right) = \begin{cases} 0, & \text{if } m \text{ is even} \\ \frac{-4}{(m\pi)^2}, & \text{if } m \text{ is odd}\end{cases}$$

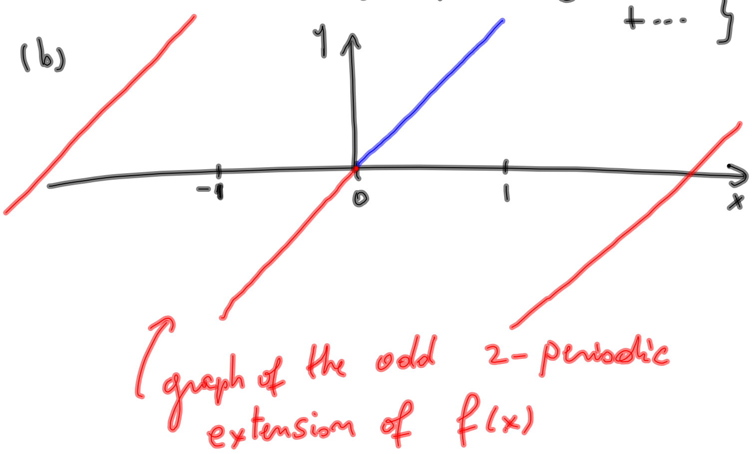
$$\therefore f(x) \simeq \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos(m\pi x)$$

$$= \frac{1}{2} + \sum_{\substack{m=1 \\ m\,\text{odd}}}^{\infty} \frac{-4}{(m\pi)^2}\cos(m\pi x)$$

$$\overset{\substack{m=1+2k \\ k\geq 0}}{=} \frac{1}{2} + \sum_{k=0}^{\infty}\frac{-4}{(1+2k)^2\pi^2}\cos\left((1+2k)\pi x\right)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left\{\frac{\cos(\pi x)}{1} + \frac{\cos(3\pi x)}{3^2} + \frac{\cos(5\pi x)}{5^2} + \dots\right\}$$

(b)



$$b_m = \frac{2}{1}\int_0^1 f(x)\sin\left(\frac{m\pi x}{1}\right)dx = 2\int_0^1 x\sin(m\pi x)\,dx$$

$$= 2\int_0^1 x\left(-\frac{\cos(m\pi x)}{m\pi}\right)'dx$$

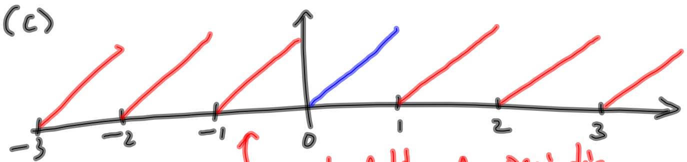
$$= 2\left\{\left[-\frac{x\cos(m\pi x)}{m\pi}\right]_0^1 - \int_0^1 1\left(-\frac{\cos(m\pi x)}{m\pi}\right)dx\right.$$

$$= 2\,\frac{(-1)^{m+1}}{m\pi} + \underbrace{\left[\frac{\sin(m\pi x)}{(m\pi)^2}\right]_0^1}_{0} = \frac{2(-1)^{m+1}}{m\pi}$$

$$\therefore\ f(x) \simeq \sum_{m=1}^{\infty} b_m \sin(m\pi x) = \sum_{m=1}^{\infty}\frac{2(-1)^{m+1}}{m\pi}\sin(m\pi x)$$

$$= \frac{2}{\pi}\left\{\frac{\sin(\pi x)}{1} - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} - \dots\right\}$$

(c)



graph of the 1-periodic extension of $f(x)$

The 1-periodic extension of $f(x)$ has the Fourier series: $\frac{C_0}{2} + \sum_{m=1}^{\infty}\left\{C_m\cos(2m\pi x) + d_m\sin(2m\pi x)\right\}$

Where $C_m = \frac{2}{1}\int_0^1 f(x)\cos(2m\pi x)\,dx = 2\int_0^1 x\cos(2m\pi x)\,dx$

$d_m = \frac{2}{1}\int_0^1 f(x)\sin(2m\pi x)\,dx = 2\int_0^1 x\sin(2m\pi x)\,dx$

We have: $C_0 = 2\int_0^1 x\,dx = 2\left[\frac{x^2}{2}\right]_0^1 = 1$

If $m \geq 1$, $C_m = 2\int_0^1 x\cos(2m\pi x)\,dx = 2\int_0^1 x\left(\frac{\sin(2m\pi x)}{2m\pi}\right)'dx$

$$= \left\{\underbrace{\left[\frac{2x\sin(2m\pi x)}{2m\pi}\right]_0^1}_{0} - 2\int_0^1 1\ \frac{\sin(2m\pi x)}{2m\pi}\,dx\right\}$$

$$= \left[\frac{\cos(2m\pi x)}{2(m\pi)^2}\right]_0^1 = 0.$$

$$d_m = 2\int_0^1 x\sin(2m\pi x)\,dx = 2\int_0^1 x\left[-\frac{\cos(2m\pi x)}{2m\pi}\right]'dx$$

$$= \left[\frac{-2x\cos(2m\pi x)}{2m\pi}\right]_0^1 + \int_0^1 1\cdot\frac{\cos(2m\pi x)}{2m\pi}\,dx$$

$$= -\frac{1}{m\pi} + \underbrace{\left[\frac{\sin(2m\pi x)}{2m\pi}\right]_0^1}_{0} = -\frac{1}{m\pi}$$

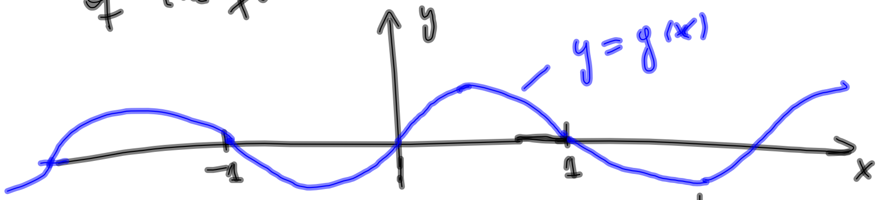
$\therefore$ The 1-periodic extension of $f(x)$ has the Fourier expansion

$$f(x) \simeq \frac{1}{2} + \sum_{m=1}^{\infty}\frac{-1}{m\pi}\sin(2m\pi x)$$

$$= \frac{1}{2} - \frac{1}{\pi}\left\{\frac{\sin(2\pi x)}{1} + \frac{\sin(4\pi x)}{2} + \frac{\sin(6\pi x)}{3} + \dots\right\}$$

Ex: Solve the IVP $\begin{cases} y'' + y = g(x) \\ y(0) = y'(0) = 0 \end{cases}$

where $g(x)$ is the odd 2-periodic extension of the function $x(1-x)$, $0 \le x \le 1$.



This could model a spring-mass system with no damping ($m=1$, $k=1$) with external force given by $g(x)$.

Sol. Idea: Expand $g(x)$ as a sine Fourier series

$$g(x) \simeq \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

where $b_n = \frac{2}{1}\int_0^1 x(1-x)\sin(n\pi x)\,dx$

$$= 2\int_0^1 x(1-x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)' dx$$

$$= \underbrace{\left[\frac{-2x(1-x)\cos(n\pi x)}{n\pi}\right]_0^1}_{0} + 2\int_0^1 (1-2x)\frac{\cos(n\pi x)}{n\pi}\,dx$$

$$= \int_0^1 (2-4x)\left(\frac{\sin(n\pi x)}{(n\pi)^2}\right)' dx$$

$$= \underbrace{\left[(2-4x)\frac{\sin(n\pi x)}{(n\pi)^2}\right]_0^1}_{0} - \int_0^1 (-4)\frac{\sin(n\pi x)}{(n\pi)^2}\,dx$$

$$= \left[\frac{-4\cos(n\pi x)}{(n\pi)^3}\right]_0^1 = \frac{-4}{(n\pi)^3}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8}{(n\pi)^3}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore\quad g(x) \simeq \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{(n\pi)^3}\sin(n\pi x)$$

To be continued...