To find a particular Solution $y_p(x)$ of $$y'' + y = \sum_{\substack{n=1\\ n\,odd}}^{\infty} \frac{8}{(n\pi)^3} \sin(n\pi x)$$

We first try to find a particular solution $y_{n,p}$ of $y'' + y = \sin(n\pi x)$, $n \geq 1$

The auxiliary eq. is $m^2 + 1 = 0$ with roots $m = \pm i$. $\therefore\ y_c(x) = C_1 \cos x + C_2 \sin x$

Using "Undetermined Coefficients", there exists a particular solution of the form:

$$y_{n,p}(x) = A_n \cos(n\pi x) + B_n \sin(n\pi x)$$

$$y''_{n,p} + y_{n,p} = A_n\left[-(n\pi)^2 + 1\right]\cos(n\pi x) + B_n\left[-(n\pi)^2 + 1\right]\sin(n\pi x) = \sin(n\pi x)$$

$$\Rightarrow A_n = 0,\quad B_n = \frac{1}{1-(n\pi)^2}$$

$$\therefore\ y_{n,p}(x) = \frac{1}{1-(n\pi)^2}\sin(n\pi x)$$

Using the superposition principle, a particular solution $y_p(x)$ of $y'' + y = \sum_{\substack{n=1\\ n\,odd}}^{\infty} \frac{8}{(n\pi)^3}\sin(n\pi x)$ is given by: $$y_p(x) = \sum_{\substack{n=1\\ n\,odd}}^{\infty} \frac{8}{(n\pi)^3(1-(n\pi)^2)}\sin(n\pi x)$$

The general solution is thus

$$y(x) = y_p(x) + y_c(x) = \sum_{\substack{n=1\\ n\,odd}}^{\infty} \frac{8}{(n\pi)^3(1-(n\pi)^2)}\sin(n\pi x) + C_1 \cos x + C_2 \sin x.$$

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y'(0) = 0,\quad y'(x) = \sum_{\substack{n=1\\ n\,odd}}^{\infty} \frac{8}{(n\pi)^2(1-(n\pi)^2)}\cos(n\pi x) + C_2 \cos x$$

$$y'(0) = 0 \Rightarrow C_2 = -\left(\sum_{\substack{n=1\\ n\,odd}}^{\infty} \frac{8}{(n\pi)^2(1-(n\pi)^2)}\right)$$

$$\therefore\ y(x) = -\left(\sum_{\substack{n=1\\ n\,odd}}^{\infty} \frac{8}{(n\pi)^2(1-(n\pi)^2)}\right)\sin x + \underbrace{\sum_{\substack{n=1\\ n\,odd}}^{\infty} \frac{8}{(n\pi)^3(1-(n\pi)^2)}\sin(n\pi x)}_{:=\ h(x)}$$

What is $h(x)$ more explicitly?

Can we compute $\sum_{n=1}^{\infty} \frac{8}{(n\pi)^2(1-(n\pi)^2)}$?

Note that $h(x)$ is solution of $y''+y = g(x)$ and $h(x)$ is 2-periodic (and odd).

In particular, $h'' + h = \begin{cases} x - x^2, & 0 \le x \le 1 \\ x + x^2, & -1 \le x \le 0 \end{cases}$

<u>On $(0,1)$</u>: A particular solution of $y''+y = x-x^2$ is of the form Ax^2+Bx+C

The general solution is

$$y = -x^2 + x + 2 + C_1 \cos x + C_2 \sin x$$

<u>On $(-1,0)$</u>: In the same way, the general solution of $y''+y = x+x^2$ is $y = x^2 + x - 2 + D_1 \cos x + D_2 \sin x$

Thus $h(x) = \begin{cases} -x^2 + x + 2 + C_1 \cos x + C_2 \sin x, & 0 \le x < 1 \\ x^2 + x - 2 + D_1 \cos x + D_2 \sin x, & -1 \le x \le 0 \end{cases}$

$h(x)$ is continuous at $x=0$:

$\Longrightarrow \quad 2 + C_1 = -2 + D_1 \Rightarrow D_1 - C_1 = 4$

$\Longrightarrow D_1 = A+2,\ C_1 = A-2$, for some A

$h'(x) = \begin{cases} -2x + 1 - C_1 \sin x + C_2 \cos x, & 0 \le x \le 1 \\ 2x + 1 - D_1 \sin x + D_2 \cos x, & -1 \le x \le 0 \end{cases}$

$h'(x)$ is continuous at 0:

$\Longrightarrow \quad 1 + C_2 = 1 + D_2 \Rightarrow C_2 = D_2 = B$

$\therefore h(x) = \begin{cases} -x^2 + x + 2 + (A-2)\cos x + B \sin x, & 0 \le x \le 1 \\ x^2 + x - 2 + (A+2)\cos x + B \sin x, & -1 \le x \le 0 \end{cases}$

To determine A, B we use the fact that $h(x)$ is 2-periodic: in particular $\underline{h(1) = h(-1)}$ and $\underline{h'(1) = h'(-1)}$

$2 + (A-2)\cos 1 + B \sin 1$

$= -2 + (A+2)\cos 1 - B \sin 1$

$\Longrightarrow 2B \sin 1 = -4 + 4\cos 1$

$B = \dfrac{2(\cos 1 - 1)}{\sin 1}$

$h'(x) = \begin{cases} -2x + 1 - (A-2)\sin x + B \cos x, & 0 \le x \le 1 \\ 2x + 1 - (A+2)\sin x + B \cos x, & -1 \le x \le 0 \end{cases}$

$h'(1) = h'(-1) \Rightarrow -1 - (A-2)\sin 1 + B\cos 1$

$= -1 + (A+2)\sin 1 + B \cos 1$

$\Longrightarrow A = 0$

$\therefore h(x) = x + \operatorname{sign}(x)\left[-x^2 + 2 - 2\cos x\right] + 2\dfrac{(\cos 1 - 1)}{\sin 1} \sin x, \quad -1 \le x \le 1$

where $\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & x < 0 \end{cases}$

$$h'(0) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{(n\pi)^2(1-(n\pi)^2)} = 1 + 2\frac{(\cos 1 - 1)}{\sin 1}$$