Ex: Expand the function e^x, $-\frac{1}{2} < x < \frac{1}{2}$ as a one-periodic Fourier series.

Sol. $p = \frac{1}{2}$

$$e^x \simeq \sum_{m \in \mathbb{Z}} c_m e^{\frac{im\pi x}{1/2}} = \sum_{m \in \mathbb{Z}} c_m e^{2im\pi x}$$

where $c_m = \frac{1}{2(\frac{1}{2})} \int_{-1/2}^{1/2} f(x) e^{-\frac{im\pi x}{1/2}} dx = \int_{-1/2}^{1/2} e^x e^{-2im\pi x} dx$

$$= \int_{-1/2}^{1/2} e^{x(1-2im\pi)} dx = \left[\frac{e^{x(1-2\pi i m)}}{1-2\pi i m} \right]_{-1/2}^{1/2}$$

$$= \frac{e^{\frac{1}{2}(1-2\pi i m)} - e^{-\frac{1}{2}(1-2\pi i m)}}{1-2\pi i m}$$

$$= \frac{e^{1/2} e^{-\pi i m} - e^{-\frac{1}{2}} e^{\pi i m}}{1-2\pi i m}$$

Since $e^{i\pi} = -1 = e^{-i\pi}$, $e^{i\pi m} = (e^{i\pi})^m = (-1)^m = e^{-i\pi m}$

$$\therefore c_m = \frac{(e^{1/2} - e^{-1/2})(-1)^m}{1-2\pi i m}$$

$$\therefore e^x \simeq (e^{1/2} - e^{-1/2}) \sum_{m \in \mathbb{Z}} \frac{(-1)^m}{1-2\pi i m} e^{2\pi i m x},$$

$$-1/2 < x < 1/2.$$

Chapter 9: Vector Calculus

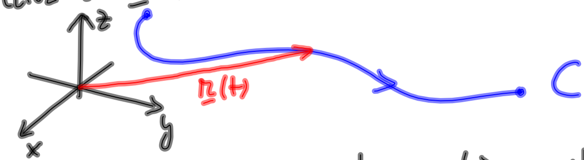
Review chapt. 7 : vectors !!

9.1 Vector functions

Def Let $I \subset \mathbb{R}$ be an interval.
A <u>vector-valued function</u> $\underline{r} : I \longrightarrow \mathbb{R}^n$ (usually $n = 2$ or 3)
is a function defined on I whose values are n-dim. vectors.
i.e. $\underline{r}(t) \in \mathbb{R}^n$ for $t \in I$

Ex: $n = 3$ $\quad \underline{r}(t) = \langle f(t), g(t), h(t) \rangle$, $t \in I$
where $f(t), g(t), h(t)$ are real-valued functions.

Rem: As t varies, the point whose position vector is $\underline{r}(t)$ describes a <u>curve</u> in $\mathbb{R}^n$.



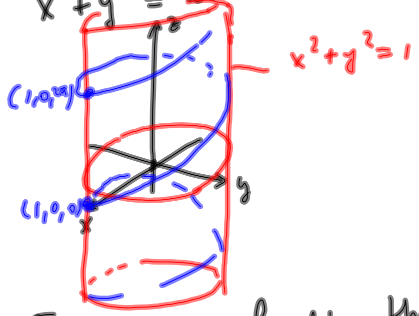
Ex: $\underline{r}(t) = \langle 2+t, 3+3t, 1+5t \rangle$, $-\infty < t < \infty$

$= \langle 2, 3, 1 \rangle + t \langle 1, 3, 5 \rangle$.

describes a line through $(2,3,1)$ with direction vector $\langle 1,3,5 \rangle$

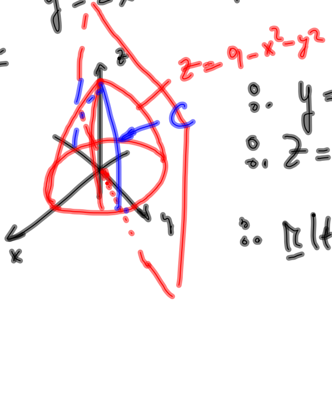
Ex: $\underline{r}(t) = \langle \cos t, \sin t, t \rangle$, $-\infty < t < \infty$

If we let $x = \cos t$, $y = \sin t$, then
$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$



Ex: Find a vector function that describes the curve C of intersection of the plane $y = 2x$ and paraboloid $z = 9 - x^2 - y^2$

Sol



We let $t = x$

$\therefore \; y = 2x = 2t$

$\therefore \; z = 9 - x^2 - y^2 = 9 - t^2 - 4t^2$
$= 9 - 5t^2$

$\therefore \; \underline{r}(t) = \langle t, 2t, 9 - 5t^2 \rangle$,
$-\infty < t < \infty$.

Def (limit) If $\underline{r}(t) = \langle f(t), g(t), h(t) \rangle$,
we write that
$$\lim_{t\to a} \underline{r}(t) = \underline{r}_0 = \langle x_0, y_0, z_0 \rangle$$
if $\lim_{t\to a} f(t) = x_0$, $\lim_{t\to a} g(t) = y_0$, $\lim_{t\to a} h(t) = z_0$.

Ex: If $\underline{r}(t) = \langle \tan t, \frac{t^7+t^3}{t^5+t^2}, \frac{\sin t}{t} \rangle$,
find $\lim_{t\to 0} \underline{r}(t)$

Sol. $\lim_{t\to 0} \tan t = 0$, $\lim_{t\to 0} \frac{t^7+t^3}{t^5+t^2} = \lim_{t\to 0} \frac{t^5+t}{t^3+1} = 0$,

$$\lim_{t\to 0} \frac{\sin t}{t} \underset{\left(\frac{0}{0}\right) \text{ l'hospital rule}}{=} \lim_{t\to 0} \frac{\cos t}{1} = 1$$

$\therefore \lim_{t\to 0} \underline{r}(t) = \langle 0, 0, 1 \rangle$. $\square$

Def The vector function $\underline{r}(t)$ is continuous at a
if $\boxed{\lim_{t\to a} \underline{r}(t) = \underline{r}(a)}$

(i.e. each component function is continuous at a)

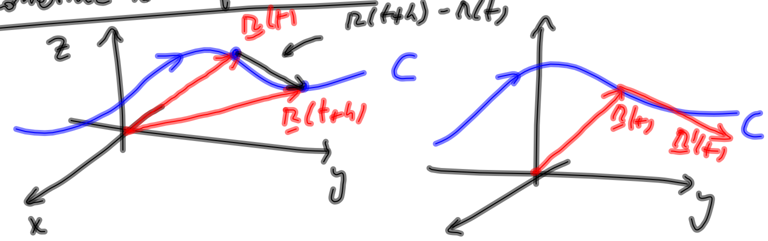
Derivatives

Def The derivative of a vector function $\underline{r}(t)$
is defined as $\boxed{\underline{r}'(t) = \lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}}$

Notation : $\underline{r}'(t) = \frac{d\underline{r}}{dt}$

Theorem: If $\underline{r}(t) = \langle f(t), g(t), h(t) \rangle$,
then $\underline{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Geometric interpretation



$\underline{r}'(t)$ is a vector tangent to C at
the point P whose position vector is $\underline{r}(t)$.