We can also define $\underline{r}''(t) = \frac{d}{dt}(\underline{r}'(t))$.

If $\underline{r}(t) = \langle f(t), g(t), h(t) \rangle$, then
$\underline{r}''(t) = \langle f''(t), g''(t), h''(t) \rangle$.

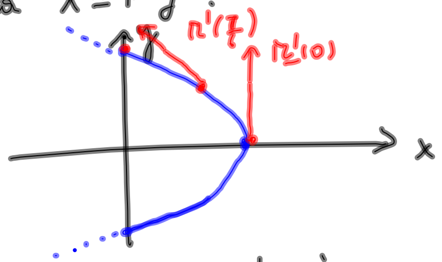
Ex: $\underline{r}(t) = \langle t^3 - 2t^2, 4t, e^{-t} \rangle$
$\underline{r}'(t) = \langle 3t^2 - 4t, 4, -e^{-t} \rangle$
$\underline{r}''(t) = \langle 6t - 4, 0, e^{-t} \rangle$.

Ex: Graph the curve traced out by the vector function $\underline{r}(t) = \frac{1+\cos(2t)}{2}\underline{i} + \sin t\, \underline{j}$, $0 \le t \le 2\pi$.

Plot $\underline{r}'(0)$, $\underline{r}'(\pi/6)$, $\underline{r}'(\pi/2)$.

Note: In 2 dim., $\underline{i} = \langle 1, 0 \rangle$, $\underline{j} = \langle 0, 1 \rangle$.
In 3 dim., $\underline{i} = \langle 1, 0, 0 \rangle$, $\underline{j} = \langle 0, 1, 0 \rangle$, $\underline{k} = \langle 0, 0, 1 \rangle$

Sol. $\frac{1+\cos(2t)}{2} = \cos^2 t = 1 - \sin^2 t$.
If $y = \sin t$, then $x = 1 - \sin^2 t = 1 - y^2$.
$\therefore$ the curve traced out by $\underline{r}(t)$ is part of the parabola $x = 1 - y^2$.



$\underline{r}'(t) = -\sin(2t)\,\underline{i} + \cos t\,\underline{j}$

$\underline{t=0}$: $\underline{r}(0) = \langle 1, 0 \rangle$, $\underline{r}'(0) = \langle 0, 1 \rangle$

$\underline{t = \pi/6}$: $\underline{r}(\pi/6) = \langle 3/4, 1/2 \rangle$
$\underline{r}'(\pi/6) = \langle -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \rangle$

$\underline{t = \pi/2}$: $\underline{r}(\pi/2) = \langle 0, 1 \rangle$
$\underline{r}'(\pi/2) = \langle 0, 0 \rangle$.

Def A vector-valued function $\underline{r}(t)$ is smooth on I if $\underline{r}'(t)$ is continuous on I and $\underline{r}'(t) \neq 0$ for all t in I. (except possibly at the end-points of I)

Ex: In the previous example, $\underline{r}(t)$ is smooth on $I = [0, \pi/4]$.
but not on the interval $I = [0, \pi]$
since $\underline{r}'(\pi/2) = 0$ and $0 < \pi/2 < \pi$.

Ex: Find the parametric equations for the line tangent to the curve C parametrized by $\underline{r}(t) = \langle t^2, t^2-t, -7t \rangle$ at the point $P = (9, 6, -21)$.

Sol. Note that $\overrightarrow{OP} = \langle 9, 6, -21 \rangle = \underline{r}(3)$.

$\underline{r}'(t) = \langle 2t, 2t-1, -7 \rangle$

$\underline{r}'(3) = \langle 6, 5, -7 \rangle$

← direction vector for the tangent line.

Tangent line has vector equation

$$\underline{s}(t) = \underline{r}(3) + t\,\underline{r}'(3) = \langle 9, 6, -21 \rangle + t \langle 6, 5, -7 \rangle$$

or:
$$\begin{cases} x = 9 + 6t \\ y = 6 + 5t \\ z = -21 - 7t \end{cases}$$

Differentiation rules

Theorem: Suppose $\underline{u}(t)$, $\underline{v}(t)$ are differentiable vector functions, c is a scalar and $f(t)$ is a real-valued function. Then,

(i) $\frac{d}{dt}\{\underline{u}(t) + \underline{v}(t)\} = \underline{u}'(t) + \underline{v}'(t)$

(ii) $\frac{d}{dt}\{c\,\underline{u}(t)\} = c\,\underline{u}'(t)$

(iii) $\frac{d}{dt}\{f(t)\,\underline{u}(t)\} = f'(t)\,\underline{u}(t) + f(t)\,\underline{u}'(t)$

(iv) $\frac{d}{dt}\{\underline{u}(t) \cdot \underline{v}(t)\} = \underline{u}'(t) \cdot \underline{v}(t) + \underline{u}(t) \cdot \underline{v}'(t)$

product rule

(v) in 3 dim., $\frac{d}{dt}\{\underline{u}(t) \times \underline{v}(t)\} = \underline{u}'(t) \times \underline{v}(t) + \underline{u}(t) \times \underline{v}'(t)$

(vi) $\frac{d}{dt}\{\underline{u}(f(t))\} = \underline{u}'(f(t)) \cdot f'(t)$ (chain rule)

Proof of (iv): $\underline{u} = \langle u_1, \dots, u_n \rangle$ (in n dim.)

$\underline{v} = \langle v_1, \dots, v_n \rangle$

$$\underline{u} \cdot \underline{v} = u_1 v_1 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

$$\frac{d}{dt}(\underline{u} \cdot \underline{v}) = \frac{d}{dt}\left(\sum_{i=1}^{n} u_i v_i\right) = \sum_{i=1}^{n} \frac{d}{dt}(u_i v_i)$$

$$= \sum_{i=1}^{n} u_i' v_i + u_i v_i' = \sum_{i=1}^{n} u_i' v_i + \sum_{i=1}^{n} u_i v_i'$$

$$= \underline{u}' \cdot \underline{v} + \underline{u} \cdot \underline{v}'.$$

Ex: Suppose $\underline{r}(t)$ describes a curve C contained in the sphere $x^2+y^2+z^2=R^2$ $(R>0)$ of radius R centered at O.

Show that $\underline{r}(t) \perp \underline{r}'(t)$.

(i.e. $\underline{r}(t) \cdot \underline{r}'(t) = 0$).

Sol. Let $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$

Then, $x(t)^2 + y(t)^2 + z(t)^2 = R^2$, for all t.

or $\|\underline{r}(t)\|^2 = \underline{r}(t) \cdot \underline{r}(t) = R^2$

By the product rule

$$\frac{d}{dt}(R^2) = 0 = \frac{d}{dt}\left(\underline{r}(t) \cdot \underline{r}(t)\right)$$

$$= \underline{r}'(t) \cdot \underline{r}(t) + \underline{r}(t) \cdot \underline{r}'(t)$$

$$= 2\, \underline{r}(t) \cdot \underline{r}'(t)$$

$$\therefore\ \underline{r}(t) \cdot \underline{r}'(t) = 0 \qquad \square$$

Integrals

Def If $\underline{r} : [a,b] \longrightarrow \mathbb{R}^3$ is a continuous vector-valued function with $\underline{r}(t) = \langle f(t), g(t), h(t) \rangle$, we define

$$\int_a^b \underline{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle.$$

Rem: If $\underline{R}(t)$ is a vector-valued function such that $\underline{R}'(t) = \underline{r}(t)$, then

$$\boxed{\int_a^b \underline{r}(t)\,dt = \underline{R}(t)\Big]_a^b = \underline{R}(b) - \underline{R}(a)}$$

Fundamental theorem of calculus.