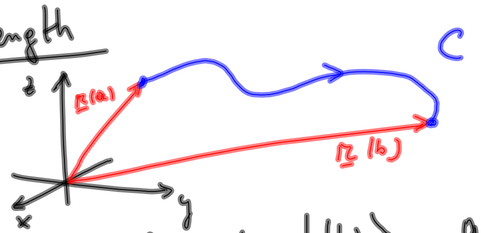
Arc length



If $\underline{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$, then the length of the curve C parametrized by $\underline{r}(t)$, $a \le t \le b$ is given by

$$\boxed{L = \int_a^b \|\underline{r}'(t)\|\,dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt}$$

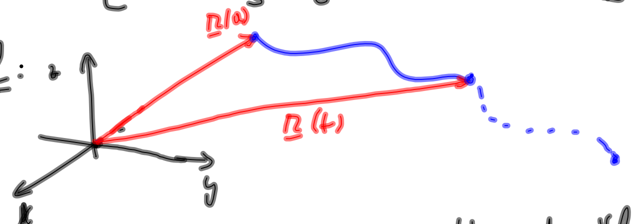
Ex: Compute the length of the curve C parametrized by $\underline{r}(t) = 3t\,\underline{i} + \sqrt{3}\,t^2\,\underline{j} + \frac{2}{3}t^3\,\underline{k}$, where $0 \le t \le 1$.

Sol. $\underline{r}'(t) = \langle 3, 2\sqrt{3}\,t, 2t^2 \rangle$

$$\|\underline{r}'(t)\| = \sqrt{3^2 + (2\sqrt{3}t)^2 + (2t^2)^2}$$
$$= \sqrt{9 + 12t^2 + 4t^4}$$
$$= \sqrt{(3 + 2t^2)^2} = 3 + 2t^2.$$

$$\therefore\ L = \int_0^1 \|\underline{r}'(t)\|\,dt = \int_0^1 3 + 2t^2\,dt$$
$$= \left[3t + \frac{2t^3}{3}\right]_0^1 = 3 + \frac{2}{3} = \frac{11}{3} \quad \square$$

Def:



If $\underline{r}(t)$, $a \le t \le b$ is a smooth vector-valued function, the corresponding arc-length function is defined by

$$s(t) = \int_a^t \|\underline{r}'(u)\|\,du, \quad a \le t \le b.$$

= length of the part of the curve C between $\underline{r}(a)$ and $\underline{r}(t)$

Rem: Any smooth curve can be "reparametrized" using the arc-length parametrization by expressing the original parameter t in terms of s.

Ex: Compute the arc-length function for the helix parametrized by
$$\underline{r}(t) = \langle b\cos t,\ b\sin t,\ ct\rangle \quad (b, c>0)$$
for $0 \le t \le 2\pi$. Reparametrize the helix using the arc-length parametrization

Sol. $\underline{r}'(t) = \langle -b\sin t,\ b\cos t,\ c\rangle$
$$\|\underline{r}'(t)\| = \sqrt{b^2\sin^2 t + b^2\cos^2 t + c^2}$$
$$= \sqrt{b^2+c^2}$$
$$\therefore\ s(t) = \int_0^t \|\underline{r}'(u)\|\,du = \int_0^t \sqrt{b^2+c^2}\,du$$
$$= \sqrt{b^2+c^2}\,t, \quad 0 \le t \le 2\pi$$
Since $t = \dfrac{s}{\sqrt{b^2+c^2}}$, the arc-length parametrization is given by:
$$\underline{r}(t(s)) = \left\langle b\cos\left(\frac{s}{\sqrt{b^2+c^2}}\right), b\sin\left(\frac{s}{\sqrt{b^2+c^2}}\right), \frac{cs}{\sqrt{b^2+c^2}}\right\rangle,$$
$$0 \le s \le 2\pi(\sqrt{b^2+c^2})$$

9.2 Motion in space

Suppose that $\underline{r}(t)$ represents the position of some object at time t. Then,
$$\underline{v}(t) = \underline{r}'(t) = \text{Velocity vector}$$
$$\underline{a}(t) = \underline{r}''(t) = \underline{v}'(t) = \text{acceleration vector}$$
$$\|\underline{v}(t)\| = \|\underline{r}'(t)\| = v(t) = \text{speed}$$

Note that $\underline{r}(t) = \underline{r}(t_0) + \int_{t_0}^{t} \underline{r}'(u)\,du$
$$= \underline{r}(t_0) + \int_{t_0}^{t} \underline{v}(u)\,du$$
$$\underline{v}(t) = \underline{v}(t_0) + \int_{t_0}^{t} \underline{v}'(u)\,du$$
$$= \underline{v}(t_0) + \int_{t_0}^{t} \underline{a}(u)\,du$$

Newton's law

If a particle of mass m has position $\underline{r}(t)$ at time t and is subjected to a force $\underline{F}$, then $\boxed{\underline{F} = m\,\underline{a}}$

Ex: Consider a particle subjected to a force whose position at time t is

$$\underline{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \le t \le \pi.$$

At time $t = \pi$, the force stops acting on the particle. Find the position of that particle at time $t = 2\pi$

Sol. For $t \le \pi$, $\underline{v}(t) = \langle -\sin t, \cos t, 1 \rangle$

$\therefore \underline{r}(\pi) = \langle -1, 0, \pi \rangle$, $\underline{v}(\pi) = \langle 0, -1, 1 \rangle$

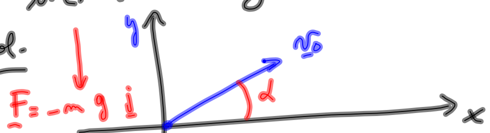
$\therefore$ If $t \ge \pi$, $\underline{v}(t) = \underline{v}(\pi) + \int_\pi^t \underbrace{\underline{a}(u)}_{0}\, du = \langle 0, -1, 1 \rangle.$

since $F = 0$ for $t > \pi$

$$\therefore \underline{r}(t) = \underline{r}(\pi) + \int_\pi^t \underline{v}(u)\, du$$
$$= \langle -1, 0, \pi \rangle + \int_\pi^t \langle 0, -1, 1 \rangle\, du$$
$$= \langle -1, 0, \pi \rangle + \Big[\langle 0, -u, u \rangle \Big]_\pi^t$$
$$= \langle -1, -(t-\pi), \pi + (t-\pi) \rangle$$
$$\therefore \underline{r}(2\pi) = \langle -1, -\pi, 2\pi \rangle. \quad \square$$

Ex: A projectile is fired with an angle of elevation α from the ground with initial velocity $\underline{v_0}$. Assuming no air resistance and that the only acting force is gravity, find the position at time t, $\underline{r}(t)$ of the projectile. What the value of α that maximizes the range (i.e. the horizontal distance travelled).

Sol.



$$\underline{F} = m\underline{a} = -mg\,\underline{j} \quad (g = 9.81\ m/sec^2)$$
$$\therefore \underline{a} = -g\,\underline{j}$$
$$\underline{v}(t) = \underbrace{\underline{v}(0)}_{\underline{v_0}} + \int_0^t \underline{a}(u)\, du = \underline{v_0} + \int_0^t -g\,\underline{j}\, du$$
$$= \underline{v_0} - gt\,\underline{j}$$
$$\underline{r}(t) = \underbrace{\underline{r}(0)}_{0} + \int_0^t \underline{v}(u)\, du = \int_0^t \underline{v_0} - gu\,\underline{j}\, du$$
$$= \underline{v_0}\, t - g\frac{t^2}{2}\,\underline{j}$$

Since $\underline{v_0} = v_0 \cos\alpha\, \underline{i} + v_0 \sin\alpha\, \underline{j}$,
where $v_0 = \|\underline{v_0}\|$,

$$\therefore \underline{r}(t) = (v_0 \cos\alpha\, t)\,\underline{i} + \left(v_0 \sin\alpha\, t - g\frac{t^2}{2}\right)\underline{j}$$

hits the ground $\Rightarrow$ y component $= 0 \Rightarrow v_0 \sin\alpha - g\frac{t}{2} = 0$

$\Rightarrow t = 2v_0 \frac{\sin\alpha}{g}$ and horizontal distance travelled in $v_0 \cos\alpha\, 2v_0 \frac{\sin\alpha}{g} = \frac{v_0^2 \sin(2\alpha)}{g}$ maximized when $\alpha = \pi/4$