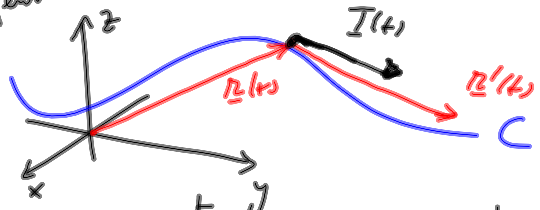
9.3 Curvature and Components of the acceleration

Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$. Note that $\mathbf{r}'(t)$ is tangent to the curve at $\mathbf{r}(t)$.



Since $s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du$, $a \le t \le b$,

we have $s'(t) = \|\mathbf{r}'(t)\|$.

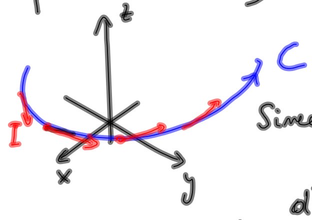
Def $T(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ = unit tangent vector (at $\mathbf{r}(t)$)

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{d\mathbf{r}}{ds}\|\mathbf{r}'(t)\|$$

chain rule $\Rightarrow \dfrac{d\mathbf{r}}{ds} = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = T(t)$

i.e. If the arc-length parametrization is used, then the corresponding tangent vector is the unit tangent vector T.

Def The curvature of C at a point P with position vector $\mathbf{r}(t)$ is $\boxed{\kappa(t) = \left\| \dfrac{dT}{ds} \right\|}$.



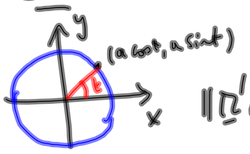
Since $\dfrac{dT}{dt} = \dfrac{dT}{ds}\underbrace{\dfrac{ds}{dt}}_{\|\mathbf{r}'\|}$

$$\Rightarrow \frac{dT}{ds} = \frac{\frac{dT}{dt}}{\|\mathbf{r}'\|}$$

$$\therefore \kappa(t) = \frac{\left\|\frac{dT}{dt}\right\|}{\|\mathbf{r}'\|}$$

Ex: Compute the curvature of a circle of radius $a > 0$.

Sol. Let $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, $0 \le t \le 2\pi$



$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a \quad (a>0)$$

$$\therefore T(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \langle -\sin t, \cos t\rangle$$

$$\frac{dT}{dt} = \langle -\cos t, -\sin t\rangle, \quad \left\|\frac{dT}{dt}\right\| = 1.$$

$$\therefore \kappa(t) = \frac{\left\|\frac{dT}{dt}\right\|}{\|\mathbf{r}'(t)\|} = \frac{1}{a}. \qquad \square$$

Theorem: If C is parametrized by $\underline{r}(t)$, then (in 3 dim.)

$$\boxed{\kappa(t) = \frac{\|\underline{r}'(t) \times \underline{r}''(t)\|}{\|\underline{r}'(t)\|^3}}$$

Recall: let $\underline{u}, \underline{v}$ be two vectors in $\mathbb{R}^3$

$0 \leq \theta \leq \pi$

θ = angle between $\underline{u}$ and $\underline{v}$

Then,

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta$$

$$\|\underline{u} \times \underline{v}\| = \|\underline{u}\| \|\underline{v}\| \sin\theta$$

Note the following: (i) $\underline{T} \times \underline{T} = \underline{0}$

(ii) $\underline{T} \perp \underline{T}'$

Proof of (ii): $\|\underline{T}\|^2 = \underline{T} \cdot \underline{T} = 1$

$$\therefore \frac{d}{dt}(\underline{T} \cdot \underline{T}) = \underline{T}' \cdot \underline{T} + \underline{T} \cdot \underline{T}' = 2\,\underline{T} \cdot \underline{T}' = \frac{d}{dt}(1) = 0$$

$$\therefore \underline{T} \cdot \underline{T}' = 0 \Rightarrow \underline{T} \perp \underline{T}'.$$

(i.e. the angle θ between $\underline{T}$ and $\underline{T}'$ is $\frac{\pi}{2}$)

$$\therefore \|\underline{T} \times \underline{T}'\| = \underbrace{\|\underline{T}\|}_{1} \|\underline{T}'\| \underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1} = \|\underline{T}'\|$$

Proof of theorem: We have $\underline{T} = \dfrac{\underline{r}'}{\|\underline{r}'\|}$

and $s(t) = \int_a^t \|\underline{r}'(u)\|\,du$, so $s'(t) = \|\underline{r}'(t)\|$.

$$\therefore \underline{r}' = \|\underline{r}'\| \underline{T} = s' \underline{T}$$

$$\therefore \underline{r}'' = s'' \underline{T} + s' \underline{T}'$$

$$\therefore \underline{r}' \times \underline{r}'' = (s' \underline{T}) \times (s'' \underline{T} + s' \underline{T}') = s' s'' \underbrace{\underline{T} \times \underline{T}}_{0} + (s')^2\, \underline{T} \times \underline{T}'$$

$$\therefore \|\underline{r}' \times \underline{r}''\| = (s')^2 \|\underline{T} \times \underline{T}'\| = (s')^2 \|\underline{T}'\|$$

$$\therefore \|\underline{T}'\| = \frac{\|\underline{r}' \times \underline{r}''\|}{(s')^2} = \frac{\|\underline{r}' \times \underline{r}''\|}{\|\underline{r}'\|^2}$$

$$\therefore \kappa(t) = \frac{\|\underline{T}'\|}{\|\underline{r}'\|} = \frac{\|\underline{r}' \times \underline{r}''\|}{\|\underline{r}'\|^3} \quad \square$$

Ex: Use the previous formula to compute the curvature of the curve C parametrized by $\underline{r}(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{3} \rangle$.

Sol.

$$\underline{r}'(t) = \langle 1, t, t^2 \rangle$$

$$\underline{r}''(t) = \langle 0, 1, 2t \rangle$$

$$\underline{r}'(t) \times \underline{r}''(t) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & t & t^2 \\ 0 & 1 & 2t \end{vmatrix} = t^2 \underline{i} - 2t \underline{j} + \underline{k}$$

$$\|\underline{r}' \times \underline{r}''\| = \sqrt{(t^2)^2 + (2t)^2 + 1} = \sqrt{1 + 4t^2 + t^4}$$

$$\|\underline{r}'\| = \sqrt{1 + t^2 + t^4}$$

$$\therefore \kappa(t) = \frac{\sqrt{1 + 4t^2 + t^4}}{(1 + t^2 + t^4)^{3/2}}$$

$\square$

Normal and binormal vectors

Since $\|\underline{T}(t)\|=1$, $\underline{T}'(t) \perp \underline{T}(t)$,

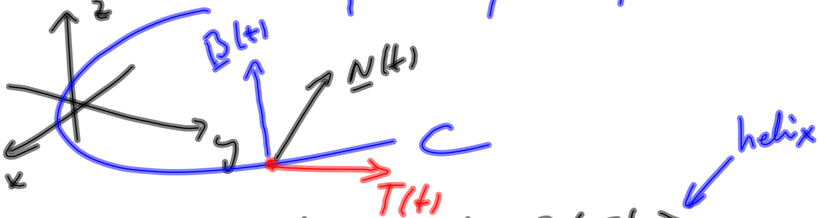
we can define $\underline{N}(t)$, the principal unit normal,

by $\boxed{\underline{N}(t) = \dfrac{\underline{T}'(t)}{\|\underline{T}'(t)\|}}$, if $\underline{T}'(t) \neq \underline{0}$

The vector $\boxed{\underline{B}(t) = \underline{T}(t) \times \underline{N}(t)}$ is called

the binormal vector. ($\underline{B}(t) \perp T(t)$ and $\underline{N}(t)$)

$$\|\underline{B}(t)\| = \underbrace{\|T(t)\|}_{1}\, \underbrace{\|\underline{N}(t)\|}_{1}\, \underbrace{\sin}_{1}(\pi/2) = 1$$



Ex: $\underline{r}(t) = \langle t, 3\sin t, 3\cos t\rangle$

$\underline{r}'(t) = \langle 1, 3\cos t, -3\sin t\rangle$

$\|\underline{r}'(t)\| = \sqrt{1 + 3^2\cos^2 t + 3^2 \sin^2 t} = \sqrt{10}$

$\underline{T}(t) = \dfrac{\underline{r}'(t)}{\|\underline{r}'(t)\|} = \langle \dfrac{1}{\sqrt{10}}, \dfrac{3}{\sqrt{10}}\cos t, \dfrac{-3}{\sqrt{10}}\sin t\rangle$

$\underline{T}'(t) = \langle 0, \dfrac{-3}{\sqrt{10}}\sin t, \dfrac{3}{\sqrt{10}}{-\cos t}\rangle$

$\| T'(t) = \dfrac{3}{\sqrt{10}}$.

$\underline{N}(t) = \dfrac{T'(t)}{\|T'(t)\|} = \langle 0, -\sin t, -\cos t\rangle$

$\therefore \underline{B}(t) = \underline{T}(t) \times \underline{N}(t) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t & \frac{-3}{\sqrt{10}}\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix}$

$= \langle -\dfrac{3}{\sqrt{10}}, \dfrac{\cos t}{\sqrt{10}}, -\dfrac{\sin t}{\sqrt{10}}\rangle$.

$$\kappa(t) = \frac{\|\underline{T}'(t)\|}{\|\underline{r}'(t)\|} = \frac{\frac{3}{\sqrt{10}}}{\sqrt{10}} = \frac{3}{10}.$$