Tangential and normal Components of the acceleration.

Let $v = \|\underline{v}\|$ = speed of a particle whose position at time is given by $\underline{r}(t)$.

Then $\underline{T}(t) = \dfrac{\underline{v}(t)}{\|\underline{v}(t)\|} = \dfrac{\underline{v}(t)}{v(t)}$

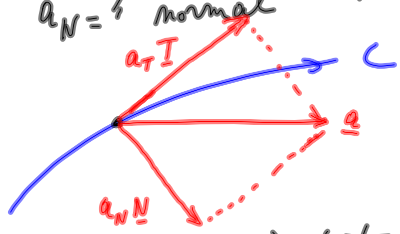
$\therefore$ $\underline{v} = v\,\underline{T}$ and $\underline{a} = \underline{v}' = v'\underline{T} + v\underline{T}'$

Since $\kappa = \dfrac{\|\underline{T}'\|}{\|\underline{r}'\|} = \dfrac{\|\underline{T}'\|}{v}$, $\|\underline{T}'\| = v\kappa$.

$\therefore$ Since $\underline{N} = \dfrac{\underline{T}'}{\|\underline{T}'\|}$, $\underline{T}' = \|\underline{T}'\|\,\underline{N} = v\kappa\,\underline{N}$

$$\therefore\quad \underline{a} = v'\underline{T} + v^2\kappa\,\underline{N} = a_T\,\underline{T} + a_N\,\underline{N}$$

where a_T = "tangential component of the acceleration"
a_N = " normal " " "



Note that $\underline{v}\cdot\underline{a} = (v\underline{T})\cdot(v'\underline{T} + v\underline{T}') = vv'\underbrace{\underline{T}\cdot\underline{T}}_{1} + v^2\underbrace{\underline{T}\cdot\underline{T}'}_{0} = vv'$

$$\therefore\quad a_T = v' = \frac{\underline{v}\cdot\underline{a}}{v} = \frac{\underline{r}'\cdot\underline{r}''}{\|\underline{r}'\|}$$

Since $\kappa = \dfrac{\|\underline{r}'\times\underline{r}''\|}{\|\underline{r}'\|^3}$,

$$a_N = v^2\kappa = \|\underline{r}'\|^2\,\frac{\|\underline{r}'\times\underline{r}''\|}{\|\underline{r}'\|^3} = \frac{\|\underline{r}'\times\underline{r}''\|}{\|\underline{r}'\|}$$

$$\therefore\quad \boxed{\begin{array}{c}\underline{a} = a_T\,\underline{T} + a_N\,\underline{N}, \text{ where} \\ a_T = \dfrac{\underline{r}'\cdot\underline{r}''}{\|\underline{r}'\|}, \quad a_N = \dfrac{\|\underline{r}'\times\underline{r}''\|}{\|\underline{r}'\|}\end{array}}$$

Ex: Find a_T and a_N if the position vector of a particle at time t is given by
$$\underline{r}(t) = \tan^{-1}t\,\underline{i} + \frac{\log(1+t^2)}{2}\,\underline{j}$$

Sol. $\underline{r}'(t) = \dfrac{1}{1+t^2}\,\underline{i} + \dfrac{t}{1+t^2}\,\underline{j}$

$\underline{r}''(t) = \dfrac{-2t}{(1+t^2)^2}\,\underline{i} + \dfrac{1-t^2}{(1+t^2)^2}\,\underline{j}$

$\|\underline{r}'(t)\| = \dfrac{1}{1+t^2}\,\|\underline{i} + t\underline{j}\| = \dfrac{\sqrt{1+t^2}}{1+t^2} = \dfrac{1}{\sqrt{1+t^2}}$

$\underline{r}'(t)\cdot\underline{r}''(t) = \dfrac{-2t}{(1+t^2)^3} + \dfrac{t-t^3}{(1+t^2)^3} = \dfrac{-t(1+t^2)}{(1+t^2)^3} = \dfrac{-t}{(1+t^2)^2}$

$$\Longrightarrow\quad a_T = \frac{\underline{r}'\cdot\underline{r}''}{\|\underline{r}'\|} = \frac{\dfrac{-t}{(1+t^2)^2}}{\dfrac{1}{\sqrt{1+t^2}}} = \frac{-t}{(1+t^2)^{3/2}}.$$

$$\underline{r}' \times \underline{r}'' = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{1}{1+t^2} & \frac{t}{1+t^2} & 0 \\ \frac{-2t}{(1+t)^2} & \frac{1-t^2}{(1+t^2)^2} & 0 \end{vmatrix} = 0\,\underline{i} + 0\,\underline{j} + \frac{1-t^2+2t^2}{(1+t^2)^3}\,\underline{k} = \frac{1+t^2}{(1+t^2)^3}\,\underline{k}$$

$$= \frac{1}{(1+t^2)^2}\,\underline{k}$$

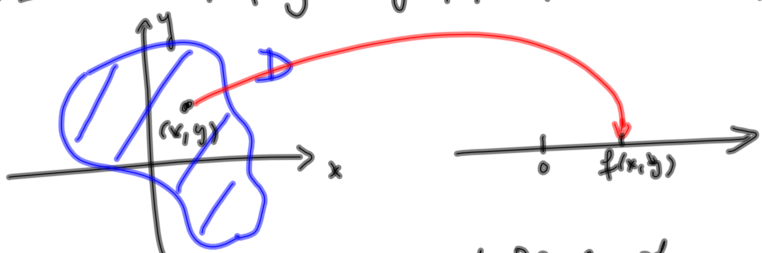
$$\|\underline{r}' \times \underline{r}''\| = \frac{1}{(1+t^2)^2}$$

$$\therefore\ a_N = \frac{\|\underline{r}' \times \underline{r}''\|}{\|\underline{r}'\|} = \frac{\frac{1}{(1+t^2)^2}}{\frac{1}{\sqrt{1+t^2}}} = \frac{1}{(1+t^2)^{3/2}} \qquad \square$$

9.4 Functions of Several Variables

Def A function of 2 variables is a rule that assigns a number $f(x,y)$ to a pair (x,y) in some domain $D \subset \mathbb{R}^2$.

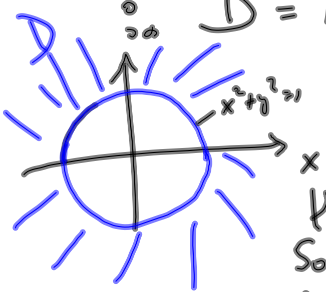
D = domain of f, range of $f = \{f(x,y),\ (x,y) \in D\}$



Ex: Find the domain and range of
$$f(x,y) = \sqrt{x^2+y^2-1}$$

Sol For $\sqrt{x^2+y^2-1}$ to be defined, we need $x^2+y^2-1 \geq 0$ or $x^2+y^2 \geq 1$

$$\therefore\ D = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2 \geq 1\}$$



If $t \geq 0$ and if we let $(x,y) = (\sqrt{1+t^2}, 0)$

then $x^2+y^2 = 1+t^2 \geq 1$

So $(x,y) \in D$

and $f(x,y) = \sqrt{x^2+y^2-1} = \sqrt{t^2} = t$ (since $t \geq 0$)

$\therefore\ t \in$ range f

Clearly, if $t < 0$, $t \in$ range f

$\therefore$ range $f = [0, \infty)$

Def If $z = f(x,y)$ is defined on D, the <u>graph of f</u> is the subset of $\mathbb{R}^3$ defined by

$$\{ (x, y, f(x,y)), \text{ where } (x,y) \in D \}$$

Ex: $f(x,y) = 3-2x-y$

the graph is the surface $z = 3-2x-2y$

or $2x + 2y + z = 3$

$\therefore$ the graph is the plane through the point $(0,0,3)$ with normal vector $\langle 2,2,1 \rangle$

Ex: Sketch the graph of $f(x,y) = \sqrt{4-x^2-y^2}$

Sol. the graph is the surface

$$z = \sqrt{4-x^2-y^2}$$

Since $z^2 = 4-x^2-y^2$, we have

$$x^2+y^2+z^2 = 4$$

(sphere centered at $(0,0,0)$ with radius)

Since $z \geq 0$, the graph is the part of that sphere <u>above</u> the x-y plane