Level curves: If $f(x,y)$ is a function of 2 variables, the level curves of $f(x,y)$ are the curves with equation $\boxed{f(x,y)=k, \ k \text{ constant}}$

Ex: $f(x,y) = xy$

the level curves have eq. $xy=k$.

If $k \neq 0$, this is the graph of $y = \frac{k}{x}$

hyperbolas

Ex: $f(x,y) = \frac{x^2}{4} + y^2$

the level curves have eq. $\frac{x^2}{4} + y^2 = k \ (k \geq 0)$

ellipses for $k>0$

Def $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

means that the values of $f(x,y)$ approach L as $(x,y) \to (x_0,y_0)$.

Ex: $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$

Sol. Since $x^2+y^2 \longrightarrow 0$ as $(x,y) \to (0,0)$

and $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

Def A function is continuous at (x_0,y_0)

if $\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).}$

Functions of 3 Variables

Let $D \subset \mathbb{R}^3$ be a subset of $\mathbb{R}^3$.

A function $f: D \longrightarrow \mathbb{R}$ is a rule that assigns a number $f(x,y,z)$ to any point (x,y,z) in D. $\text{dom } f = D = \{(x,y,z), \text{ where } f(x,y,z) \text{ is defined}\}$.

Ex: $f(x,y,z) = \frac{\ln(x^2+y^2-1)}{z}$

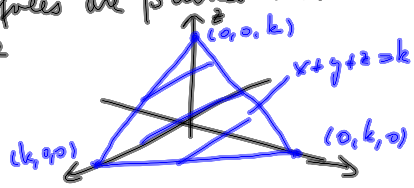
$\text{dom } f = \{(x,y,z), \ x^2+y^2-1>0, \ z \neq 0\}$

= region outside of the cylinder $x^2+y^2=1$ and not on the x-y plane.

Def: A level surface of a function $f(x,y,z)$ is a surface with equation $f(x,y,z)=k$, k a constant

Ex: $f(x,y,z) = x+y+z$

The level surfaces are planes with eq. $x+y+z=k$



Partial derivatives

Def If $f(x,y)$ is a function of 2 variables, we define

- $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$
  = partial derivative of f with respect to x at (a,b)
- $f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$
  = partial derivative of f with respect to y at (a,b).

Notation: If $z=f(x,y)$, $f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$

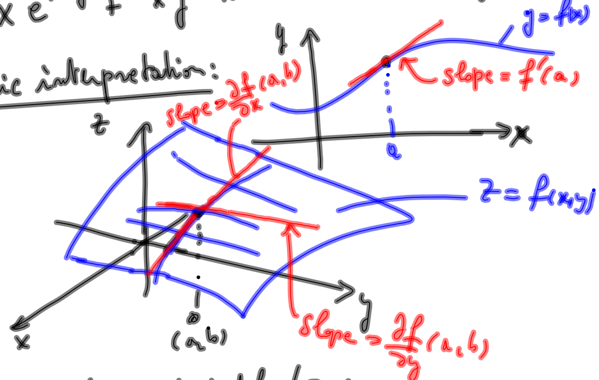
$$f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

Ex: $f(x,y) = xy\,e^{x^2y}$. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Sol. $\frac{\partial f}{\partial x} = y e^{x^2y} + xy\,(2xy)e^{x^2y} = y(1+2x^2y)e^{x^2y}$

$\frac{\partial f}{\partial y} = x e^{x^2y} + xy\;x^2 e^{x^2y} = x(1+x^2y)e^{x^2y}$

Geometric interpretation:



Functions of 3 variables (or more)

If $w=f(x,y,z)$, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are defined in a similar way.

Ex: $f(x,y,z) = x\,e^{xyz}$

$\frac{\partial f}{\partial x} = e^{xyz} + x(yz)e^{xyz} = (1+xyz)e^{xyz}$

$\frac{\partial f}{\partial y} = x(xz)e^{xyz} = x^2z\,e^{xyz}$

$\frac{\partial f}{\partial z} = x(xy)e^{xyz} = x^2y\,e^{xyz}$

Higher order partial derivatives

If $z=f(x,y)$, we can define

$f_{xx} = (f_x)_x$, $f_{yx} = (f_y)_x$, $f_{xy} = (f_x)_y$, $f_{yy} = (f_y)_y$

2nd order partials of $f(x,y)$

Notation: $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

Ex: $f(x,y) = x \ln(x+y)$.

Compute the 1st and 2nd order partials.

Sol.

$$\frac{\partial f}{\partial x} = \ln(x+y) + \frac{x}{x+y}$$

$$\frac{\partial f}{\partial y} = \frac{x}{x+y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{1}{x+y} + \frac{x+y-x}{(x+y)^2} = \frac{x+2y}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{1}{x+y} - \frac{x}{(x+y)^2} = \frac{y}{(x+y)^2}$$

//

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{x+y-x}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -\frac{x}{(x+y)^2}$$

Theorem (Clairaut): Equality of mixed partials

If f_{xy} and f_{yx} are continuous, then

$$\boxed{f_{xy} = f_{yx}.}$$