The chain rule

1-Variable Case: $f(x)$, $x = x(t)$

$$\boxed{\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}}$$

Theorem (case I): Suppose $z = f(x,y)$ is a differentiable function of x and y and $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then,

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Ex: Let $z = x e^{xy}$, $x = t^2$, $y = t^3$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (e^{xy} + xye^{xy})(2t) + (x^2e^{xy})(3t^2)$$

$$= (e^{t^5} + t^5e^{t^5})(2t) + (t^4e^{t^5})(3t^2)$$

$$= e^{t^5}(2t + 5t^6).$$

Theorem (case II): Suppose z is a differentiable function of x and y and $x = g(s,t)$, $y = h(s,t)$ are differentiable functions of s and t. Then,

$$\boxed{\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\end{aligned}}$$

Theorem (case III): Suppose u is a differentiable function of $x_1, x_2, \dots, x_m$ and each variable x_i, $i = 1, \dots, m$ is itself a differentiable function of $t_1, \dots, t_n$. Then, u viewed as a function of $t_1, \dots, t_n$ satisfies:

$$\boxed{\begin{aligned}\frac{\partial u}{\partial t_1} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \dots + \frac{\partial u}{\partial x_m}\frac{\partial x_m}{\partial t_1}\\ &\vdots\\ \frac{\partial u}{\partial t_n} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_n} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_n} + \dots + \frac{\partial u}{\partial x_m}\frac{\partial x_m}{\partial t_n}\end{aligned}}$$

Ex: Suppose $z = x^2 + xy$, $x = e^t s$, $y = (\sin s)t^2$. Compute $\frac{\partial z}{\partial t}$, $\frac{\partial z}{\partial s}$ using the chain rule.

Sol. $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2x+y)(e^t s) + x(2t\sin s)$$

$$= (2e^t s + \sin s\, t^2)(e^t s) + e^t s(2t\sin s)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x+y)(e^t) + x((\cos s)t^2)$$

$$= (2e^t s + \sin s\, t^2)e^t + (e^t s)((\cos s)t^2)$$

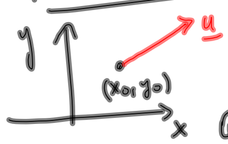
Ex: let $w = xyz$ where $x = uv$, $y = ut$ and $z = tv$. Compute $\frac{\partial w}{\partial u}$ using the chain rule.

Sol. $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= (yz)(v) + (xz)(t) + (xy)(0)$$

$$= (ut^2 v)(v) + (uv^2 t)t = 2uv^2t^2.$$ □

9.5 Directional derivatives and the gradient vector.



$\underline{u} = \langle a, b\rangle$ unit vector

$f(x,y)$ given function

Question: What is the rate of change of $f(x,y)$ at (x_0,y_0) in the direction of $\underline{u}$?

Def: If $z = f(x,y)$, the directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of $\underline{u} = \langle a,b\rangle$ ($\underline{u}$ unit vector) is

$$\boxed{(D_{\underline{u}} f)(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}}$$

if limit exists.

Ex: • $\underline{u} = \langle 1,0\rangle = \underline{i}$

$$(D_{\underline{i}} f)(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} = \frac{\partial f}{\partial x}(x_0,y_0)$$

• $\underline{u} = \langle -1, 0\rangle = -\underline{i}$

$$(D_{-\underline{i}} f)(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0-h, y_0) - f(x_0,y_0)}{h}$$

$$= \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0,y_0)}{-h} = -\frac{\partial f}{\partial x}(x_0,y_0)$$

Similarly, $(D_{\underline{j}} f)(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0)$

$(D_{-\underline{j}} f)(x_0,y_0) = -\frac{\partial f}{\partial y}(x_0,y_0).$

Theorem: If $z = f(x,y)$ is differentiable and $\underline{u} = \langle a,b\rangle$ is a unit vector, then

$$(D_{\underline{u}} f)(x_0,y_0) = f_x(x_0,y_0)\, a + f_y(x_0,y_0)\, b$$

$$= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \langle a,b\rangle$$

Def $\boxed{\nabla f(x_0,y_0) := \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle}$ is called the gradient of f at (x_0,y_0).

$$\therefore \quad \boxed{D_{\underline{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot u} \quad (*)$$

Proof of (*): let $g(h) = f(x_0+ha, y_0+hb)$. Then,

$$g'(0) = \lim_{h\to 0}\frac{g(h) - g(0)}{h} = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h} = D_{\underline{u}} f(x_0,y_0)$$

let $x = x_0 + ha$, $y = y_0 + hb$. Then

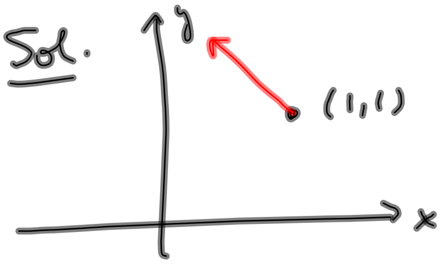
$$g'(0) = g'(h)\Big|_{h=0} = \frac{\partial f}{\partial x}(x_0,y_0)\, a + \frac{\partial f}{\partial y}(x_0,y_0)\, b$$

chain rule

□

Ex: Compute the directional derivative of $f(x,y) = x^2 y$ at $(1,1)$ in the direction of the vector $\langle -1, 1 \rangle$.

Sol.



A unit vector $\underline{u}$ in the same direction as $\langle -1,1\rangle$ is $\underline{u} = -\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$\nabla f(x,y) = \langle 2xy, x^2 \rangle$

$\nabla f(1,1) = \langle 2,1 \rangle$.

$$D_{\underline{u}} f(1,1) = \nabla f(1,1) \cdot \underline{u} = \langle 2,1\rangle \cdot \langle -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Functions of 3 Variables

Def If $w = f(x,y,z)$ is a differentiable function of 3 variables, then the gradient of f at (x_0,y_0,z_0) is defined by

$$\boxed{\nabla f(x_0,y_0,z_0) = \langle \frac{\partial f}{\partial x}(x_0,y_0,z_0), \frac{\partial f}{\partial y}(x_0,y_0,z_0), \frac{\partial f}{\partial z}(x_0,y_0,z_0) \rangle}$$

Def Direction derivative : $\underline{u}$ unit vector, $\underline{u} = \langle a,b,c \rangle$

$$\boxed{D_{\underline{u}} f(x_0,y_0,z) = \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}}$$

Theorem: If $z = f(x,y,z)$ is differentiable and $\underline{u}$ is a unit vector, we have

$$\boxed{D_{\underline{u}} f(x_0,y_0,z_0) = \nabla f(x_0,y_0,z_0) \cdot u}$$