Ex: Compute the directional derivative of
$f(x,y,z) = xy^2 - 4x^2y + z^2$ at $(1,-1,2)$
in the direction of the vector $6\underline{i} + 2\underline{j} + 3\underline{k}$.

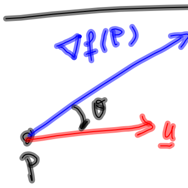
Sol. $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle y^2 - 8xy, 2xy - 4x^2, 2z \rangle$

$\nabla f(1,-1,2) = \langle 9, -6, 4 \rangle$. $\quad \|\langle 6,2,3\rangle\| = \sqrt{6^2+2^2+3^2} = \sqrt{49} = 7$

A unit vector with the same direction as $\langle 6,2,3\rangle$ is $\langle \frac{6}{7}, \frac{2}{7}, \frac{3}{7} \rangle$.

$$(D_{\underline{u}}f)(1,-1,2) = \nabla f(1,-1,2) \cdot \underline{u} = \langle 9,-6,4\rangle \cdot \langle \frac{6}{7}, \frac{2}{7}, \frac{3}{7}\rangle$$
$$= \frac{54 - 12 + 12}{7} = \frac{54}{7}. \quad \square$$

Direction of maximum (and minimum) rate of change



$(D_{\underline{u}}f)(P) = \nabla f(P) \cdot \underline{u} = \|\nabla f(P)\| \underbrace{\|\underline{u}\|}_{1} \cos\theta$
$= \|\nabla f(P)\| \cos\theta,$
where θ is the angle between $\nabla f(P)$ and $\underline{u}$

$\therefore$ If we let $\underline{u}$ vary (with P fixed), $(D_{\underline{u}}f)(P)$ is:

- Maximized when $\cos\theta = 1$, i.e. $\theta = 0$
  i.e. when $\underline{u}$ has the same direction as $\nabla f(P)$
- Minimized when $\cos\theta = -1$, i.e. $\theta = \pi$
  i.e. $\underline{u}$ has direction opposite to that of the gradient vector.

max. possible value of $D_{\underline{u}}f(P) = \|\nabla f(P)\|$

Min. " " " " $= -\|\nabla(P)\|$.

Ex: In which direction does the function $f(x,y,z) = \frac{x-z}{y+z}$ increases the fastest at $(-1,1,3)$?
What is the maximum rate of change of $f(x,y,z)$ at that point?

Sol. $\nabla f(x,y,z) = \langle \frac{1}{y+z}, \frac{z-x}{(y+z)^2}, \frac{-(y+z)-(x-z)}{y+z} \rangle$

$\nabla f(-1,1,3) = \langle \frac{1}{4}, \frac{4}{4^2}, 0 \rangle = \langle \frac{1}{4}, \frac{1}{4}, 0 \rangle$

$\therefore f(x,y,z)$ increase the fastest in the direction of $\langle \frac{1}{4}, \frac{1}{4}, 0\rangle$ (or that of $\underline{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$).

The maximum rate of change of $f(x,y,z)$ at $(-1,1,3)$ is

$$\|\nabla f(-1,1,3)\| = \|\langle \frac{1}{4}, \frac{1}{4}, 0\rangle\| = \frac{\sqrt{2}}{4} \quad \square$$

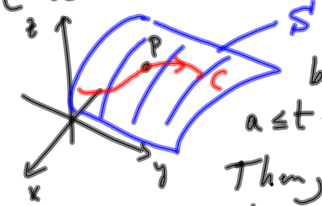
9.6 Tangent planes to level surfaces

Consider a level S with equation

$$\boxed{F(x,y,z)=k}\ ,$$

where $F(x,y,z)$ is differentiable and k is a constant.

Let $P=(x_0,y_0,z_0)$ be a point on S and let C be a curve on C passing through P



If C is parametrized by $\underline{r}(t)=\langle x(t),y(t),z(t)\rangle$, $a\le t\le b$ and $\underline{r}(t_0)=\overrightarrow{OP}$.

Then, $F(x(t),y(t),z(t))=k$

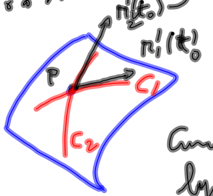
$\therefore$ Then, using the chain rule we have

$$\frac{d}{dt}\left\{F(x(t),y(t),z(t))\right\}=\frac{d}{dt}k=0$$

or $\quad F_x(\cdot)\,x'(t)+F_y(\cdot)\,y'(t)+F_z(\cdot)\,z'(t)=0$

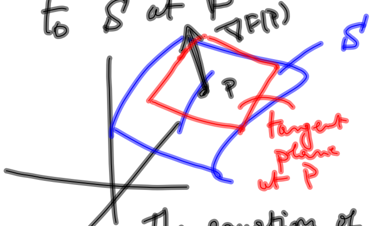
$$\therefore\ \boxed{\nabla F(\underline{r}(t))\cdot\underline{r}'(t)=0}$$

$\therefore$ At $t=t_0$, $\quad \nabla F(P)\cdot\underline{r}'(t)=0$



$\therefore$ $\nabla F(P)$ is $\perp$ to the tangent vector at P of any curve C passing through P and lying in S.

$\therefore$ $\nabla F(P)\perp$ to the tangent plane to S at P



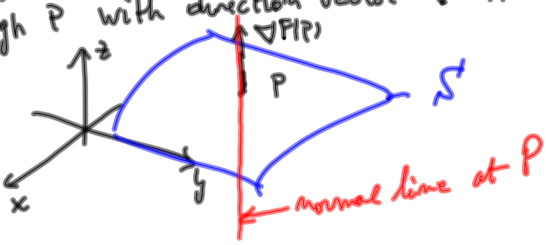
The equation of the plane tangent to S at P is thus:

$$\boxed{\nabla F(P)\cdot(\underline{r}-\underline{r}_0)=0}\ ,$$

where $\underline{r}=\langle x,y,z\rangle$, $\underline{r}_0=\overrightarrow{OP}=\langle x_0,y_0,z_0\rangle$

or $$\boxed{\frac{\partial F}{\partial x}(P)(x-x_0)+\frac{\partial F}{\partial y}(P)(y-y_0)+\frac{\partial F}{\partial z}(z-z_0)=0.}$$

Def: the normal line to the surface $F(x,y,z)=k$ at P (P on S) is the line through P with direction vector $\nabla F(P)$.



Ex: Find the equation of the plane tangent to the sphere $x^2+y^2+z^2=4$ at the point $P=(-1,1,\sqrt{2})$.

Sol. Take $F(x,y,z) = x^2+y^2+z^2$

$$\nabla F(x,y,z) = \langle 2x, 2y, 2z \rangle$$

$$\nabla F(-1,1,\sqrt{2}) = \langle -2, 2, 2\sqrt{2} \rangle$$

$\therefore$ the equation of the tangent plane at $(-1,1,\sqrt{2})$ is thus $(-2)(x-(-1)) + 2(y-1) + 2\sqrt{2}(z-\sqrt{2}) = 0$

or $\quad -x+y+\sqrt{2}z = 4$ $\quad \square$

Rem: In dim. 2, if $F(x,y)=k$ is a level curve and P is a point on the curve, then $\nabla F(P)$ is $\perp$ to the curve at P