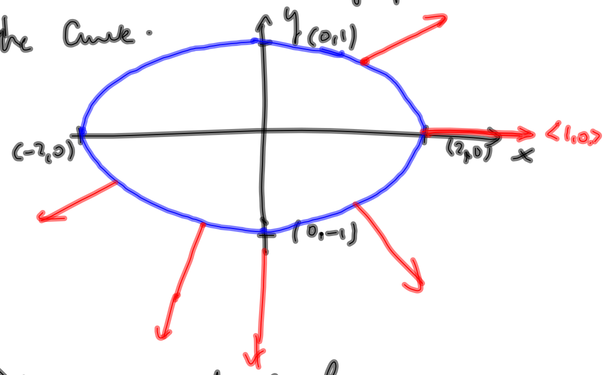
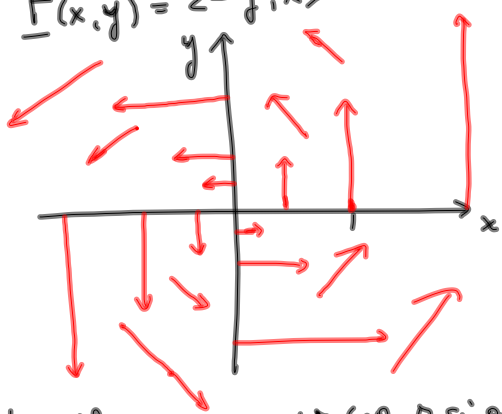
Ex: The curve $\frac{x^2}{4} + y^2 = 1$ (Ellipse) is a level curve curve of $F(x,y) = \frac{x^2}{4} + y^2$. The vector $\nabla F = \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle = \langle \frac{x}{2}, 2y \rangle$ is $\perp$ to the curve at any point (x,y) on the curve.



9.7 Divergence and curl.

Def A vector field defined on a set $U \subset \mathbb{R}^n$ is a function $\underline{F} : U \to \mathbb{R}^n$.

Ex: $\underline{F}(x,y) = \langle -y, x \rangle$



Note that if $\langle x, y \rangle = \langle r\cos\theta, r\sin\theta \rangle$, then $\langle r\cos(\theta + \frac{\pi}{2}), r\sin(\theta + \frac{\pi}{2}) \rangle$

$$= \langle -r\sin\theta, r\cos\theta \rangle = \langle -y, x \rangle$$

Ex: Gravitational vector field.

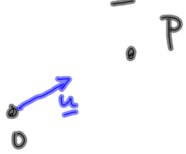
Suppose a mass M located at $(0,0,0)$. The gravitational force due to M on a mass m located at $P = (x,y,z)$ is

$$\underline{F}(x,y,z) = -\frac{GMm}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle,$$

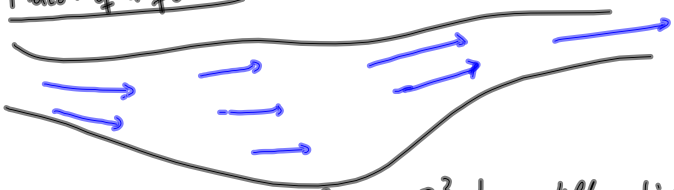
or, if $\underline{r} = \langle x, y, z \rangle$,

$$\underline{F}(\underline{r}) = -\frac{GMm\,\underline{r}}{\|\underline{r}\|^3} = -\frac{GMm}{\|\underline{r}\|^2} \underbrace{\frac{\underline{r}}{\|\underline{r}\|}}_{\underline{u}}$$

$\underline{u}$ = unit vector pointing from O to P.



Motion of a fluid.



Def let $\underline{F}: \mho \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field with $\underline{F} = \langle F_1, F_2, F_3 \rangle$, then

We define $$\boxed{\operatorname{div}(\underline{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

= divergence of $\underline{F}$ (scalar function)

and

$$\boxed{\operatorname{Curl}(\underline{F}) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k}}$$

= Curl of F (vector field)

The operator ∇ ("del" operator) is defined by $$\boxed{\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}}.$$

If $f: \mho \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then
$$\nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}. \quad (\text{gradient of } f)$$

Formally, if $\underline{F}$ is a vector field,
$$\nabla \cdot \underline{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle$$
$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \operatorname{div}(\underline{F}).$$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k} = \operatorname{Curl}(\underline{F})$$

$$\therefore \boxed{\nabla \cdot \underline{F} = \operatorname{div}(\underline{F}), \ \nabla \cdot \underline{F} = \operatorname{Curl}(F).}$$

Ex: $\underline{F}(x,y,z) = \langle 3xy + z, \sqrt{z} + y, \frac{z}{y} \rangle$

$$\nabla \cdot \underline{F} = \frac{\partial}{\partial x}(3xy + z) + \frac{\partial}{\partial y}(\sqrt{z} + y) + \frac{\partial}{\partial z}(z/y)$$
$$= 3y + 1 + \frac{1}{y}$$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy + z & \sqrt{z} + y & \frac{z}{y} \end{vmatrix}$$
$$= \left(-\frac{z}{y^2} + \frac{1}{2\sqrt{z}}\right)\underline{i} + (1 - 0)\underline{j} + (0 - 3x)\underline{k}$$
$$= \left(-\frac{z}{y^2} + \frac{1}{2\sqrt{z}}\right)\underline{i} + \underline{j} - 3x\underline{k}$$

Theorem: (i) curl(grad f) $= \underline{0}$
(ii) div(curl $\underline{F}$) $= 0$

Formally, $\nabla \times \nabla f = \underline{0}$, $\nabla \cdot (\nabla \times \underline{F}) = 0$

Proof (i) $\nabla \times (\nabla f) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\underline{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\underline{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial x}\right)\underline{k} = \underline{0}$$

using equality of mixed partials

(ii) $\nabla \cdot (\nabla \times \underline{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

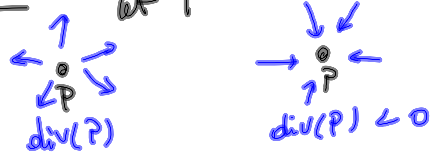
$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0$$

Interpretation.

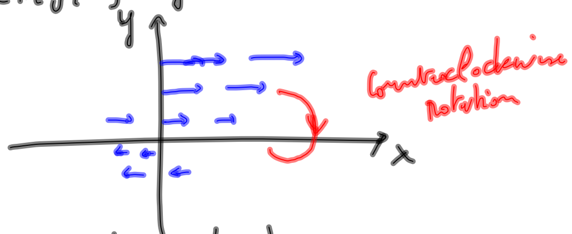
Suppose $\underline{F}$ represents the velocity vector in a fluid

divergence: total outflow per unit of volume at P



Curl F: $(\nabla \times \underline{F})(P)$ measure the "tendency" of the fluid to "swirl" around P.

Ex: $\underline{F}(x, y, z) = y\,\underline{i}$



$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\underline{k}$$