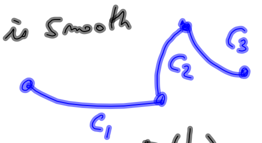
9.8 line integrals

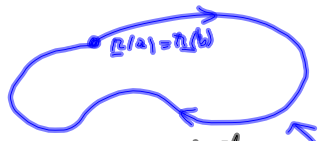
let C be a curve parametrized by $\underline{r}(t)$, $a \le t \le b$.

- C is <u>smooth</u> if $\underline{r}'(t)$ is continuous and $\underline{r}'(t) \neq 0$, for $a < t < b$.

- C is <u>piecewise smooth</u> if $C = C_1 \cup C_2 \cup \dots C_m$, where each curve C_i is smooth

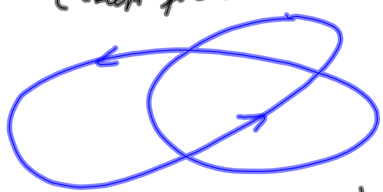


- C is <u>closed</u> if $\underline{r}(a) = \underline{r}(b)$

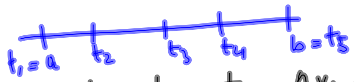


- C is a simple closed curve if it is closed and does not intersect itself (except for $t = a$ and $t = b$)

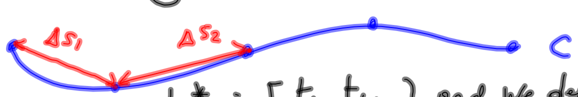
Simple closed curve



<u>Def.</u> let C be parametrized by $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b$. If P is a partition of $[a,b]$ given by points $a = t_1 < t_2 < \dots < b = t_m$, we let $\|P\| = \max_i |t_{i+1} - t_i|$ = length of longest subinterval



$\Delta t_k = t_{k+1} - t_k$, $\Delta x_k = x_{k+1} - x_k$, $\Delta y_k = y_{k+1} - y_k$,
$\Delta z_k = z_{k+1} - z_k$, $\Delta S_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2}$



We choose points t_k^* in $[t_k, t_{k+1}]$ and we define
$x_k^* = x(t_k^*)$, $y_k^* = y(t_k^*)$, $z_k^* = z(t_k^*)$

If $G(x,y,z)$ is a function defined and continuous on C, we define:

(1) $\int_C G(x,y,z)\, dx = \lim_{\|P\| \to 0} \sum_{k=1}^{m} G(x_k^*, y_k^*, z_k^*)\, \Delta x_k$

(2) $\quad$ " $\quad dy =$ $\quad$ " $\quad$ " $\quad$ " $\quad \Delta y_k$

(3) $\quad$ " $\quad dz =$ $\quad$ " $\quad$ " $\quad$ " $\quad \Delta z_k$

(4) $\int_C G(x,y,z)\, ds =$ $\quad$ " $\quad$ " $\quad$ " $\quad \Delta S_k$,

arc-length

if the limit exists.

Proposition: If C is parametrized by $\underline{r}(t)$, $a \le t \le b$ and C is piecewise smooth, then

(1) $\int_C G(x,y,z)\,dx = \int_a^b G(x(t),y(t),z(t))\; x'(t)\,dt$

(2) $\int_C G(x,y,z)\,dy =$ " $\quad y'(t)$ "

(3) $\int_C G(x,y,z)\,dz =$ " $\quad z'(t)$ "

(4) $\int_C G(x,y,z)\,ds = \int_a^b G(x(t),y(t),z(t))\,\|\underline{r}'(t)\|\,dt$

Rem: If $G \equiv 1$, $\boxed{\int_C 1\,ds = \text{length of } C.}$

Ex: Let C be parametrized by
$\underline{r}(t) = \langle a\cos t, a\sin t, bt \rangle$, $0 \le t \le 2\pi$
$(a,b>0)$. Compute $\int_C \frac{z}{x^2+y^2+z^2}\,dz$, $\int_C \frac{1}{x^2+y^2+z^2}\,ds$

Sol. $\underline{r}'(t) = \langle -a\sin t, a\cos t, b \rangle$

$\|\underline{r}'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$

$$\int_C \frac{z}{x^2+y^2+z^2}\,dz = \int_0^{2\pi} \frac{bt}{a^2+b^2t^2}\, b\,dt = b^2\int_0^{2\pi}\frac{t}{a^2+b^2t^2}\,dt$$

$$= b^2\left[\frac{\ln(a^2+b^2t^2)}{2b^2}\right]_0^{2\pi} = \frac{\ln(a^2+4\pi^2b^2)-\ln(a^2)}{2}$$

$$= \frac{1}{2}\ln\left(\frac{a^2+4\pi^2b^2}{a^2}\right)$$

$$\int_C \frac{1}{x^2+y^2+z^2}\,ds = \int_0^{2\pi}\frac{1}{a^2+b^2t^2}\sqrt{a^2+b^2}\,dt$$

$$= \frac{\sqrt{a^2+b^2}}{a^2}\int_0^{2\pi}\frac{1}{1+\left(\frac{bt}{a}\right)^2}\,dt$$

$$= \frac{\sqrt{a^2+b^2}}{ba}\left[\tan^{-1}\left(\frac{b}{a}t\right)\right]_0^{2\pi} = \frac{\sqrt{a^2+b^2}}{ba}\tan^{-1}\left(\frac{2\pi b}{a}\right) \quad \square$$

Ex: Evaluate $\int_C y\,dx + x\,dy + z\,dz$, where C is parametrized by $\underline{r}(t) = \langle \cos t, \sin t, t^2 \rangle$, $0 \le t \le 2\pi$.

Sol. $\underline{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle -\sin t, \cos t, 2t \rangle$

$$\int_C y\,dx + x\,dy + z\,dz = \int_0^{2\pi} (\sin t)(-\sin t) + (\cos t)(\cos t) + t^2(2t)\,dt$$

$$= \int_0^{2\pi} \cos^2 t - \sin^2 t + 2t^3\,dt = \int_0^{2\pi} \cos(2t) + 2t^3\,dt$$

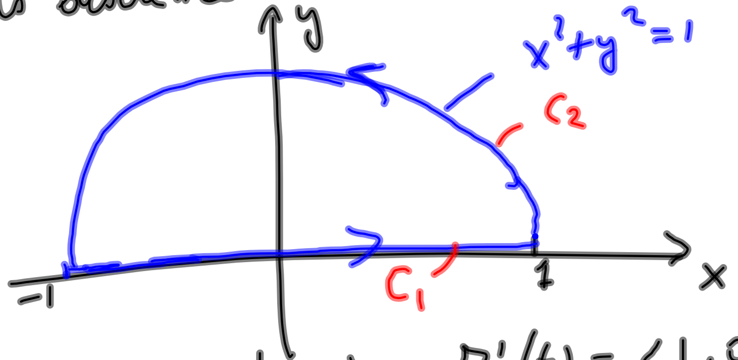
$$= \left[\frac{\sin(2t)}{2} + 2\frac{t^4}{4}\right]_0^{2\pi} = \frac{(2\pi)^4}{2} = 8\pi^4 \quad \square$$

Notation: If C is a closed curve, we will write:

$$\int_C P\,dx + Q\,dy + R\,dz = \oint_C P\,dx + Q\,dy + R\,dz$$

"circulation"

Ex: Compute $\oint_C xy\,dx + x^2\,dy$, where the curve C is described below:



Sol.: **On C_1:** $\underline{r}(t) = \langle t, 0 \rangle$, $\underline{r}'(t) = \langle 1, 0 \rangle$

$-1 \le t \le 1$, $x'(t) = 1$, $y'(t) = 0$

$$\int_{C_1} xy\,dx + x^2\,dy = \int_{-1}^{1} (t)(0)(1) + t^2(0)\,dt = 0$$

On C_2: $\underline{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$.

$\underline{r}'(t) = \langle -\sin t, \cos t \rangle$

$$\int_{C_2} xy\,dx + x^2\,dy = \int_0^{\pi} (\cos t)(\sin t)(-\sin t) + (\cos^2 t)\cos t\,dt$$

$$= \int_0^{\pi} -\cos t \sin^2 t + \cos t \cos^2 t\,dt$$

$$= \int_0^{\pi} \cos t\,[1 - 2\sin^2 t]\,dt$$

$$= \left[\sin t - 2\frac{\sin^3 t}{3} \right]_0^{\pi} = 0.$$

$$\int_C xy\,dx + x^2\,dy = \int_{C_1} \dots + \int_{C_2} \dots = 0 + 0 = 0$$

$\square$