Def If C is a piecewise smooth curve and
$\underline{F}(x,y,z) = P(x,y,z)\,\underline{i} + Q(x,y,z)\,\underline{j} + R(x,y,z)\,\underline{k}$
is a continuous vector field, we define

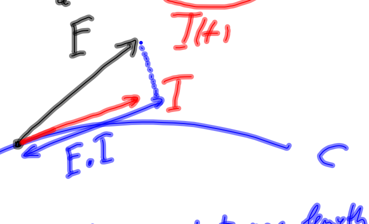
$$\boxed{\int_C \underline{F}\cdot d\underline{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.}$$

Rem: If C is parametrized by $\underline{r}(t)$, $a \le t \le b$, $\underline{r}(t) = \langle x(t), y(t), z(t)\rangle$
we have

$$\int_C \underline{F}\cdot d\underline{r} = \int_a^b P(\underline{r}(t))\,x'(t) + Q(\underline{r}(t))\,y'(t) + R(\underline{r}(t))\,z'(t)\,dt$$

$$= \int_a^b \underline{F}(\underline{r}(t))\cdot \underline{r}'(t)\,dt = \int_a^b \underline{F}(\underline{r}(t))\cdot \underbrace{\frac{\underline{r}'(t)}{\|\underline{r}'(t)\|}}_{T(t)}\,\|\underline{r}'(t)\|\,dt$$

$$= \int_C \underline{F}\cdot T\,ds$$



= integral with respect to arc-length of the tangential component of $\underline{F}$ along C

Rem: In physics, if $\underline{F}(x,y,z)$ is a force vector field (e.g. gravity), then

$$\int_C \underline{F}\cdot d\underline{r} = W = \text{Work done by the force along the path } C$$

Ex: Compute the work done by the force field $\underline{F}(x,y,z) = (y-z)\,\underline{i} + (z-x)\,\underline{j} + (x-y)\,\underline{k}$ on a particle moving on the helix parametrized by $\underline{r}(t) = \langle a\cos t, a\sin t, bt\rangle$, $0 \le t \le 2\pi$ $(a,b>0)$

Sol: $\underline{r}'(t) = \langle -a\sin t, a\cos t, b\rangle$

$$W = \int_C \underline{F}\cdot d\underline{r} = \int_0^{2\pi} \underline{F}(\underline{r}(t))\cdot \underline{r}'(t)\,dt$$

$$= \int_0^{2\pi} \langle a\sin t - bt,\ bt - a\cos t,\ a\cos t - a\sin t\rangle \cdot \langle -a\sin t, a\cos t, b\rangle\,dt$$

$$= \int_0^{2\pi} -a^2\sin^2 t + abt\sin t + abt\cos t - a^2\cos^2 t + ab\cos t - ab\sin t\,dt$$

$$= \int_0^{2\pi} -a^2 + ab(t\sin t + t\cos t) + ab(\cos t - \sin t)\,dt$$

$$\int_0^{2\pi} t(\sin t + \cos t)\,dt = \int_0^{2\pi} t(-\cos t + \sin t)'\,dt$$

$$= \Big[t(-\cos t + \sin t)\Big]_0^{2\pi} - \underbrace{\int_0^{2\pi} -\cos t + \sin t\,dt}_{0}$$

$$\int_0^{2\pi}\cos t\,dt = \int_0^{2\pi}\sin t\,dt = 0$$

$$= -2\pi$$

$$\Rightarrow W = -a^2\,2\pi + ab(-2\pi) = -2\pi a(a+b)$$

Work = gain in kinetic energy

Consider a particle moving along the trajectory $\underline{r}(t)$, $a \le t \le b$ from A to B (with $\overrightarrow{OA} = \underline{r}(a)$, $\overrightarrow{OB} = \underline{r}(b)$), under the influence of a force field $\underline{F}(x,y,z)$

Under Newton 2nd law, $\underline{F} = m\,\underline{a} = m\,\underline{r}''$

$$\therefore W = \int_C \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\,dt$$

$$= \int_a^b m\,\underline{r}''(t) \cdot \underline{r}'(t)\,dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt}\left\{\underline{r}'(t) \cdot \underline{r}'(t)\right\} dt$$

$$= \frac{m}{2}\Big[\,\|\underline{r}'(t)\|^2\Big]_{t=a}^{t=b} = \underbrace{\frac{m\,\|\underline{r}'(b)\|^2}{2} - \frac{m\,\|\underline{r}'(a)\|^2}{2}}_{\text{change in kinetic energy of the particle going from A to B.}}$$

9.9. Line integrals independent of paths

2 Variables

Def If $\phi(x,y)$ is a differentiable function, we define the differential of $\phi(x,y)$ is the formal expression $\boxed{d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy}$

- An expression $P(x,y)\,dx + Q(x,y)\,dy$ is called an exact differential if there exists a function $\phi(x,y)$ such that $d\phi = P\,dx + Q\,dy$, i.e. $P = \frac{\partial \phi}{\partial x}$, $Q = \frac{\partial \phi}{\partial y}$.

Theorem: (Fundamental theorem of line integrals)

Given 2 functions $P(x,y)$ and $Q(x,y)$ such that there exists a function $\phi(x,y)$ such that $d\phi = P\,dx + Q\,dy$, then

$$\boxed{\int_C P(x,y)\,dx + Q(x,y)\,dy = \phi(B) - \phi(A),}$$

where A is the initial point of C and B the ending pt.

Proof Let C be parametrized by $\underline{r}(t)$, $a \le t \le b$.
We have $\underline{r}(a) = \overrightarrow{OA}$, $\underline{r}(b) = \overrightarrow{OB}$.

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b P(\underline{r}(t))\,x'(t) + Q(\underline{r}(t))\,y'(t)\,dt$$

$$= \int_a^b \frac{\partial \phi}{\partial x}(x(t),y(t))\,x'(t) + \frac{\partial \phi}{\partial y}(x(t),y(t))\,y'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left\{ \phi(x(t),y(t)) \right\} dt$$

chain rule

$$= \phi(\underline{r}(b)) - \phi(\underline{r}(a))$$

FTC

$$= \phi(B) - \phi(A) \qquad \square$$

Consequence: If $P(x,y)\,dx + Q(x,y)\,dy$ is exact, then

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = 0$$

if C is closed path (since $A = B$ in that case).