Rem: If $P(x,y)\,dx + Q(x,y)\,dy$ is exact and $d\phi = P\,dx + Q\,dy$, then, letting $\underline{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$, then

$$\underline{F}(x,y) = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle = \nabla \phi$$

$$\therefore \int_C \underline{F} \cdot d\underline{r} = \int_C P\,dx + Q\,dy = \phi(B) - \phi(A),$$

where A is the beginning pt of C
B " " ending " " "

integral is "independent of path".

If this is the case, we call the vector field $\underline{F}$ conservative and $\phi(x,y)$ is called a potential function for $\underline{F}$.

- In Physics, if $\underline{F}$ represents a force field and $\underline{F}$ is conservative, i.e. there exists a potential function $\phi(x,y)$ such that $\nabla\phi = \underline{F}$, then the function $p(x,y) = -\phi(x,y)$ is called the potential energy function and the total energy of a particle moving under the influence of the conservative force vector field $\underline{F}$ is given by

$$\boxed{E(t) = \underbrace{p(\underline{r}(t))}_{\text{potential energy}} + \tfrac{1}{2} m \underbrace{\|\underline{r}'(t)\|^2}_{\text{kinetic energy}}}$$

The total energy is conserved:

$$\frac{d}{dt}\{E(t)\} = \frac{d}{dt}\left\{-\phi(\underline{r}(t)) + \tfrac{1}{2} m\, \underline{r}'(t)\cdot\underline{r}'(t)\right\}$$

$$= -\nabla\phi(\underline{r}(t)) \cdot \underline{r}'(t) + \frac{m}{2}\, 2\,\underline{r}'(t)\cdot\underline{r}''(t)$$

$$= \Big[\underbrace{-\underline{F}(\underline{r}(t)) + m\,\underline{r}''(t)}_{\substack{0 \\ \text{by Newton's 2nd law}}}\Big] \cdot \underline{r}'(t) = 0$$

Def. • A set $\mathcal{U} \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) is open if $\mathcal{U}$ contains none of its boundary points

$\mathcal{U}$ is not open $\qquad$ $\mathcal{U}$ is open

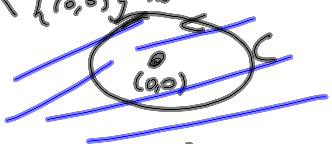
• An open $\mathcal{U} \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) is connected if any points A, B in $\mathcal{U}$ can be joined by a path C which is entirely contained in $\mathcal{U}$.
(i.e. $\mathcal{U}$ is in "one piece")

$\mathcal{U}$ is not connected $\qquad$ $\mathcal{U}$ is connected

• An open set $\mathcal{U}$ is simply connected if
(a) $\mathcal{U}$ is connected
(b) Any closed curve C lying inside $\mathcal{U}$ can be continuously deformed to a point in $\mathcal{U}$ while staying inside of $\mathcal{U}$.

$\mathcal{U}$ not simply connected $\qquad$ $\mathcal{U}$ simply connected

$\mathcal{U} = \mathbb{R}^2$ is simply connected
$\mathcal{U} = \mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected



$\mathcal{U} = \mathbb{R}^2 \setminus \{[0,\infty) \times \{0\}\}$ is simply connected

$\mathcal{U} = \mathbb{R}^3$ is simply connected
$\mathcal{U} = \mathbb{R}^3 \setminus \{(0,0,0)\}$ is simply connected.

Rem: If $F = \langle P, Q \rangle$ and $P = \frac{\partial \phi}{\partial x}$ and $Q = \frac{\partial \phi}{\partial y}$ for some potential function $\phi(x,y)$, then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial Q}{\partial x}.$$

$\Rightarrow$ a necessary condition for the existence of a potential function $\phi(x,y)$: $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$.

Theorem: Let $P(x,y)$, $Q(x,y)$ have continuous 1^{st} order partials defined on some simply connected open set $D \subset \mathbb{R}^2$. Then, the following are equivalent:

(a) $\int_C P\,dx + Q\,dy$ is independent of path (in D)

(b) There exists a function $\phi(x,y)$ such that $\nabla\phi = \underline{F}$, where $\underline{F} = \langle P, Q\rangle$ (or $d\phi = P\,dx + Q\,dy$).

(c) $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } D.$$

Ex: Let $\underline{F}(x,y) = (x^2 - 2xy)\,\underline{i} + (y^2 - x^2)\,\underline{j}$

(i) Show that $\underline{F}$ is conservative on $\mathbb{R}^2$ and find a potential function for $\underline{F}$ on $\mathbb{R}^2$.

(ii) Compute $\int_C \underline{F}\cdot d\underline{r}$, where C is any path starting at $(0,0)$ and ending at $(1,1)$.

Sol. $P(x,y) = x^2 - 2xy$, $Q(x,y) = y^2 - x^2$ are defined (as well as their 1^{st} order partials) and continuous on $\mathbb{R}^2$ which is simply connected. We have:

$$\frac{\partial P}{\partial y} = -2x = \frac{\partial Q}{\partial x}$$

$\therefore$ By the previous (with $D = \mathbb{R}^2$), $\underline{F} = \langle P, Q\rangle$ is conservative, i.e. there exists a potential function $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = x^2 - 2xy \quad (1)$$

$$\frac{\partial \phi}{\partial y} = y^2 - x^2 \quad (2)$$

From (1), $\phi(x,y) = \frac{x^3}{3} - x^2 y + C(y)$

$$\therefore \frac{\partial \phi}{\partial y} = -x^2 + C'(y) \underset{(2)}{=} y^2 - x^2$$

$$\Rightarrow C'(y) = y^2 \Rightarrow C(y) = \frac{y^3}{3} + c$$

$$\Rightarrow \phi(x,y) = \frac{x^3}{3} - x^2 y + \frac{y^3}{3}$$

(taking $c = 0$)

(b) $$\int_C \underline{F}\cdot d\underline{r} = \phi(1,1) - \phi(0,0)$$
$$= \left(\frac{1}{3} - 1 + \frac{1}{3}\right) - 0 = -\frac{1}{3}$$

$\square$