3 variables

- $\phi(x,y,z) \longrightarrow d\phi = \underbrace{\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz}_{\text{differential of } \phi}$

- A differential form $P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$ is exact if there exist a function $\phi(x,y,z)$ such that $d\phi = P\,dx + Q\,dy + R\,dz$ (1)
(i.e. $\nabla \phi = \langle P, Q, R \rangle$).

- If $P\,dx + Q\,dy + R\,dz$ is exact and $\underline{F} = \langle P, Q, R \rangle$, then

$$\int_C \underline{F} \cdot d\underline{r} = \int_C P\,dx + Q\,dy + R\,dz = \phi(B) - \phi(A),$$

where A is the beginning pt of C,
B " " ending " ".

Theorem: let P, Q, R have continuous 1st order partials on some open, simply connected set $U \subset \mathbb{R}^3$.
Then, the following are equivalent
(a) The differential form $P\,dx + Q\,dy + R\,dz$ is exact
(b) The integrals $\int_C P\,dx + Q\,dy + R\,dz$ ($\int_C \underline{F} \cdot d\underline{r}$ where $\underline{F} = \langle P, Q, R \rangle$) are independent of path.
(c) $\nabla \times \underline{F} = \underline{0}$, if $\underline{F} = \langle P, Q, R \rangle$. (2)

Rem: (2) is necessary since if $\underline{F} = \nabla \phi$, then $\nabla \times \underline{F} = \nabla \times \nabla \phi = \underline{0}$.

Ex: Show that the integrals
$$\int_C 2x\,dx + (z^2 + 2yz)\,dy + (y^2 + 2yz)\,dz$$
are independent of path and compute this integral if C is any path starting at $(0,0,0)$ and ending at $(1,1,1)$.

Sol: let $\underline{F} = \langle 2x, z^2 + 2yz, y^2 + 2yz \rangle$.

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & z^2 + 2yz & y^2 + 2yz \end{vmatrix}$$
$$= \underline{i}\left[(2y + 2z) - (2z + 2y)\right] + \underline{j}\left[0 - 0\right] + \underline{k}\left[0 - 0\right] = \underline{0}.$$

$\therefore$ $\underline{F}$ is conservative, i.e. there exists a function $\phi(x,y,z)$ with $\underline{F} = \nabla \phi$.

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = z^2 + 2yz, \quad \frac{\partial \phi}{\partial z} = y^2 + 2yz$$
(i) (ii) (iii)

From (i), $\phi(x,y,z) = x^2 + C(y,z)$

$$\therefore \quad \frac{\partial \phi}{\partial y} = \frac{\partial C}{\partial y} \overset{(ii)}{=} z^2 + 2yz$$

$$\therefore \quad C(y,z) = yz^2 + y^2 z + D(z)$$

$$\therefore \quad \phi(x,y,z) = x^2 + yz^2 + y^2 z + D(z)$$

$$\therefore \quad \frac{\partial \phi}{\partial z} = 2yz + y^2 + D'(z) \overset{(iii)}{=} y^2 + 2yz$$

$$\therefore \quad D'(z) = 0, \text{ i.e. } \underbrace{D(z) = D_0}_{\text{a constant}}$$

Taking $D_0 = 0$, we obtain

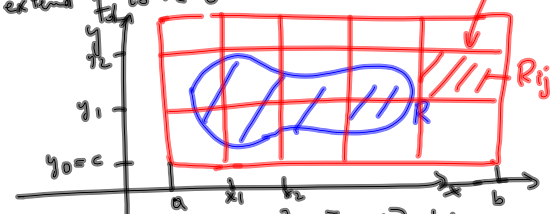
$$\phi(x,y,z) = x^2 + yz^2 + y^2 z.$$

$$\int_C \underline{F} \cdot d\underline{r} = \phi(1,1,1) - \phi(0,0,0) = 3 - 0 = 3$$

D

9.10 Double integrals

Consider a function $f(x,y)$ defined on some closed and bounded region $R \subset \mathbb{R}^2$.
We choose a rectangle $[a,b] \times [c,d] \supset R$ and we extend f to be 0 outside of R.

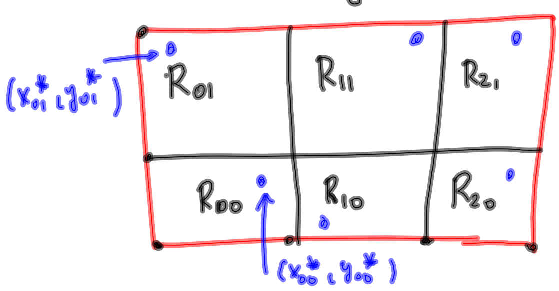


Given a partition P of $[a,b] \times [c,d]$ defined by $a = x_0 < x_1 < \ldots < x_m = b$, $c = y_0 < y_1 < \ldots < y_m = d$,
we consider the subrectangles $[x_{i-1}, x_i] \times [y_{j-1}, y_j] := R_{ij}$
and choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}.
Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $\Delta A_{ij} = \Delta x_i \Delta y_j$.
We then form the sum

$$\sum_{i,j} f(x_{ij}^*, y_{ij}^*) \, \Delta A_{ij},$$

and define $\displaystyle\iint_R f(x,y)\, dA = \lim_{\|P\| \to 0} \sum_{i,j} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$,

where $\|P\| = \max_{i,j} \{\Delta x_i, \Delta y_j\}$



– If the integral of $f(x,y)$ exists, then f is called integrable on R.

Fact: If $f(x,y)$ is continuous on R, then $f(x,y)$ is integrable on R.

Rem: (a) $\boxed{\iint_R 1 \, dA = \text{Area of } R}$

(b) If $z = f(x,y) \geq 0$ for $(x,y) \in R$, then the integral $\iint_R f(x,y)\, dA$ is the Volume of the solid region below the graph of $f(x,y)$ and above the x-y plane, for (x,y) in R.