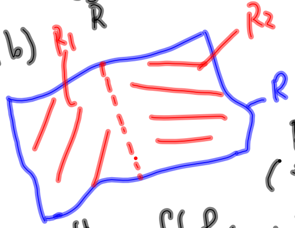
Properties

(a) If $f(x,y)$ and $g(x,y)$ are integrable on $\mathbb{R}$, then so is $\alpha f(x,y) + \beta g(x,y)$ $(\alpha, \beta \in \mathbb{R})$

and $\displaystyle\iint_R \alpha f(x,y) + \beta g(x,y)\, dA = \alpha \iint_R f(x,y)\, dA + \beta \iint_R g(x,y)\, dA$

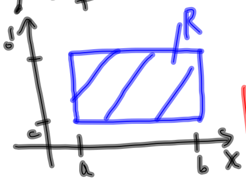
(b) 

If $R = R_1 \cup R_2$, where the 2 subregions R_1 and R_2 do not overlap (except perhaps on their boundary)

then $\displaystyle\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$

(c) (Fubini's theorem)

If $R = [a,b] \times [c,d] = \{(x,y),\ a \le x \le b,\ c \le y \le d\}$, and f is integrable on R,



then

$$\boxed{\iint_R f(x,y)\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy.}$$

Ex: Compute $\displaystyle\iint_R x\sqrt{x^2+y}\, dA$ where $R = [0,1] \times [0,3]$.

Sol. 1st way $\displaystyle\iint_R x\sqrt{x^2+y}\, dA = \int_0^1 \left(\int_0^3 x\sqrt{x^2+y}\, dy \right) dx$

$\displaystyle = \int_0^1 \left[\tfrac{2}{3} x (x^2+y)^{3/2} \right]_{y=0}^{y=3} dx = \int_0^1 \tfrac{2}{3} x (x^2+3)^{3/2} - \tfrac{2}{3} x^4\, dx$

$\displaystyle = \left[\tfrac{2}{15} (x^2+3)^{5/2} - \tfrac{2}{15} x^5 \right]_0^1 = \tfrac{2}{15}\left[(32-1) - 3^{5/2} \right] = \frac{62 - 18\sqrt{3}}{15}.$

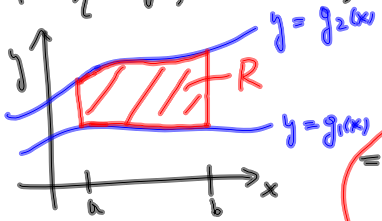
2nd way:

$\displaystyle\iint_R x\sqrt{x^2+y}\, dA = \int_0^3 \left(\int_0^1 x\sqrt{x^2+y}\, dx \right) dy$

$\displaystyle = \cdots = \frac{62}{15}$ (check!)

Regions of type I : region between the graphs of 2 functions of x.

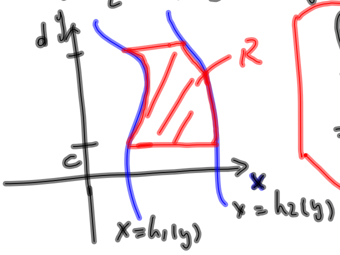
$R = \{(x,y),\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$



$$\boxed{\iint_R f(x,y)\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx}$$

Regions of type I : regions between the graphs of 2 functions of y

$$R=\{(x,y),\ c\le y\le d,\ h_1(y)\le x\le h_2(y)\}.$$

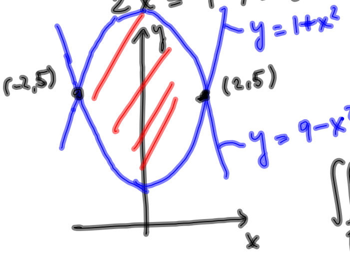


$$\boxed{\iint_R f(x,y)\,dA = \int_c^d\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy}$$

Ex: Evaluate $\iint_R 1+x^2\,dA$ where R is the region bounded by the graphs of $y=1+x^2$ and $y=9-x^2$.

Sol. The 2 curves intersect when $1+x^2=9-x^2$

$2x^2=9-1=8 \implies x^2=4$ i.e. $x=\pm 2$. $y=5$



$$R=\{(x,y),\ -2\le x\le 2,\ 1+x^2\le y\le 9-x^2\}$$

$$\iint_R 1+x^2\,dx=\int_{-2}^{2}\left(\int_{1+x^2}^{9-x^2} 1+x^2\,dy\right)dx$$

$$=\int_{-2}^{2}(1+x^2)(8-2x^2)\,dx=\int_{-2}^{2}8+6x^2-2x^4\,dx$$

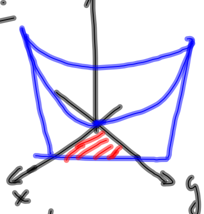
$$=2\int_0^2 8+6x^2-2x^4\,dx=2\left[8x+\frac{6x^3}{3}-2\frac{x^5}{5}\right]_0^2$$

$$=2\left(16+16-\frac{64}{5}\right)=2\left(\frac{96}{5}\right)=\frac{192}{5}$$

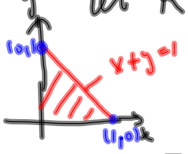
$\square$

Ex: Find the volume of the solid region in the 1st octant $(x,y,z\ge 0)$ bounded by the paraboloid $z=x^2+y^2$, the plane $x+y=1$ and the coordinate planes

Sol.



The solid region is the region below the graph of $z=x^2+y^2$ and above the triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$ on the x-y plane.



let $R=\{(x,y),\ 0\le x\le 1,\ 0\le y\le 1-x\}$

$$V=\iint_R x^2+y^2\,dA=\int_0^1\left(\int_0^{1-x} x^2+y^2\,dy\right)dx$$

$$=\int_0^1\left[x^2y+\frac{y^3}{3}\right]_0^{1-x}dx=\int_0^1 x^2(1-x)+\frac{(1-x)^3}{3}\,dx$$

$$=\int_0^1 x^2-x^3+\frac{(1-x)^3}{3}\,dx=\left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{(1-x)^4}{12}\right]_0^1$$

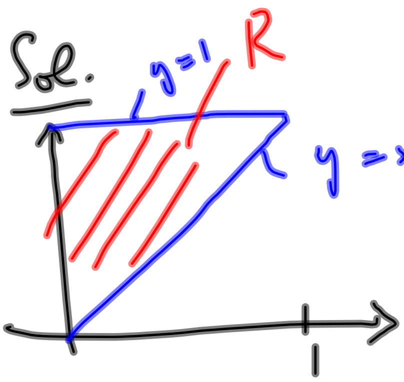
$$=\frac{1}{3}-\frac{1}{4}+\frac{1}{12}=\frac{1}{12}+\frac{1}{12}=\frac{1}{6}\quad\square$$

Ex: Evaluate $\int_0^1 \left(\int_x^1 \sin(y^2)\, dy \right) dx$

$$\| $$

$$\iint_R \sin(y^2)\, dA$$

Sol.



As a region of type II,

$$R = \{ (x,y),\ 0 \le y \le 1,\ 0 \le x \le y \}.$$

$$\iint_R \sin y^2\, dA = \int_0^1 \left(\int_0^y \sin y^2\, dx \right) dy$$

$$= \int_0^1 y \sin y^2\, dy = \left[-\frac{\cos(y^2)}{2} \right]_0^1$$

$$= \frac{1-\cos(1)}{2}. \qquad \square$$