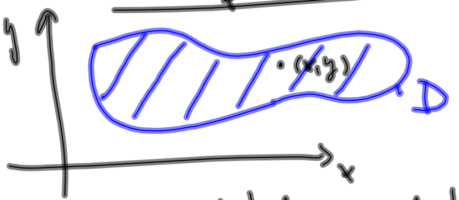
Lamina with variable mass density — Center of mass.



$\rho(x,y)$ mass density per unit of area at (x,y)
$(\rho(x,y) \geq 0)$

Def: • total mass of the lamina: $m = \iint_D \rho(x,y)\,dA$

• moment about the y-axis: $M_y = \iint_D x\,\rho(x,y)\,dA$

• moment about the x-axis: $M_x = \iint_D y\,\rho(x,y)\,dA$

• Center of mass: $(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

$$= \left(\frac{1}{m}\iint_D x\,\rho(x,y)\,dA,\ \frac{1}{m}\iint_D y\,\rho(x,y)\,dA\right)$$

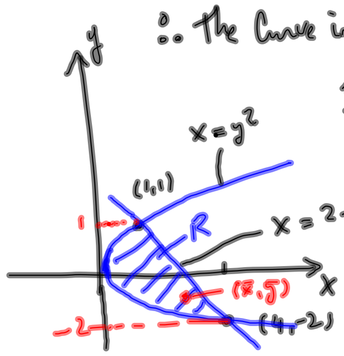
• moments of inertia: $I_x = \iint_D y^2 \rho(x,y)\,dA$

$$I_y = \iint_D x^2 \rho(x,y)\,dA$$

Ex: Find the center of mass of the lamina occupying the region R bounded by the graphs of $x = y^2$ and $x = 2-y$ if $\rho(x,y) = x$.

Sol: We have $y^2 = 2-y$ or $y^2 + y - 2 = 0$
or $(y-1)(y+2) = 0$

$\therefore$ The curve intersect at $(1,1)$ and $(4,-2)$.



$R = \{(x,y),\ -2 \leq y \leq 1,\ y^2 \leq x \leq 2-y\}$

$m = \iint_R \rho(x,y)\,dA$

$$= \int_{-2}^{1}\int_{y^2}^{2-y} x\,dx\,dy = \frac{36}{5}$$

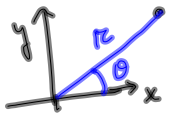
Similarly,

$$M_y = \iint_R x\,\rho(x,y)\,dA = \int_{-2}^{1}\int_{y^2}^{2-y} x^2\,dA = \frac{423}{28}$$

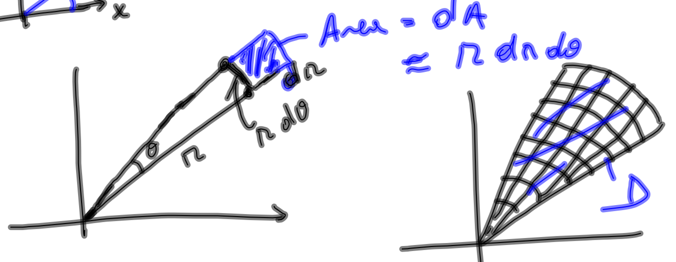
$$M_x = \iint_R y\,\rho(x,y)\,dA = \int_{-2}^{1}\int_{y^2}^{2-y} xy\,dA = -\frac{45}{8}$$

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{235}{112}, \frac{-25}{32}\right) \approx (2.09, -0.78)$$

9.11 Double integrals in polar coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad r = \sqrt{x^2+y^2}, \quad \tan\theta = y/x$$



$$\iint_D f(x,y)\,dA \approx \underbrace{\sum_i f(r_i\cos\theta_i, r_i\sin\theta_i)\, r_i\,\Delta r_i\,\Delta\theta_i}_{\text{Riemann Sum}}$$

for $\iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

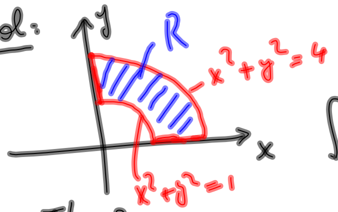
Where D^* is the region D expressed in polar coord.

Theorem: If $f(x,y)$ is continuous on a region D in the plane and D is expressed as the region D^* in polar coordinates (i.e. in the r,θ plane), then

$$\boxed{dA = r\,dr\,d\theta} \qquad \boxed{\iint_D f(x,y)\,dA = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}.$$

Ex: Suppose R is the region in the 1^{st} quadrant ($x,y \geq 0$) bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$. Compute $\iint_R 3x+y^2\,dA$ using polar coordinates.

Sol:



$R^* = \{(r,\theta),\ 1 \leq r \leq 2,\ 0 \leq \theta \leq \frac{\pi}{2}\}$

$$\iint_R 3x+y^2\,dA = \iint_{R^*} (3r\cos\theta + r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_1^2 3r^2\cos\theta + r^3\sin^2\theta \,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[r^3\cos\theta + \frac{r^4}{4}\sin^2\theta \right]_{r=1}^{r=2} d\theta$$

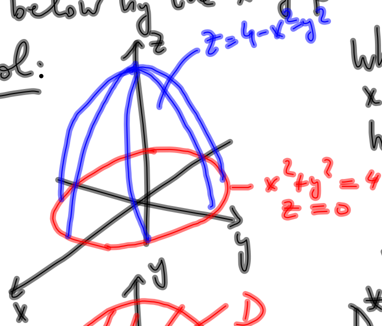
$$= \int_0^{\pi/2} 7\cos\theta + \frac{15}{4}\underbrace{\sin^2\theta}_{\frac{1-\cos(2\theta)}{2}}\,d\theta$$

$$= \left[7\sin\theta + \frac{15}{8}\theta - \frac{15}{16}\sin(2\theta) \right]_0^{\pi/2}$$

$$= 7 + \frac{15\pi}{16} \qquad \square$$

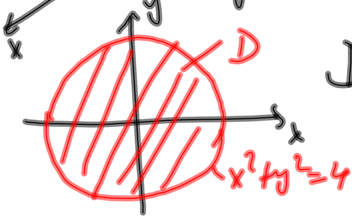
Ex: Let E be the solid region bounded above by the paraboloid $z = 4-x^2-y^2$ and below by the x-y plane. Find the volume of E.

Sol:



When $z=0$, i.e. on the x,y plane, the paraboloid has equation $x^2+y^2=4$.

$$D=\{(x,y),\ x^2+y^2\le 4\}$$



$$D^*=\{(r,\theta),\ 0\le r\le 2,\ 0\le\theta\le 2\pi\}$$

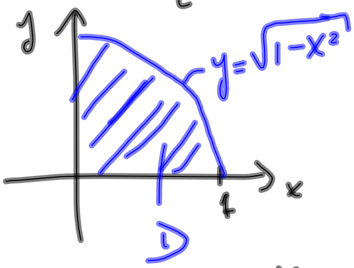
$$\therefore \operatorname{Vol}(E) = \iint_D 4-x^2-y^2\,dA = \iint_{D^*}(4-r^2)\,r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^2 4r-r^3\,dr\,d\theta = \int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} 8-4\,d\theta = 8\pi \qquad \square$$

Ex: Evaluate $I = \int_0^1\int_0^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}}\,dy\,dx$

Sol: $D=\{(x,y),\ 0\le x\le 1,\ 0\le y\le\sqrt{1-x^2}\}$



$$D^*=\{(r,\theta),\ 0\le r\le 1,\ 0\le\theta\le\tfrac{\pi}{2}\}$$

$$I = \iint_{D^*} e^r\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r e^r\,dr\,d\theta$$

$$= \frac{\pi}{2}\int_0^1 r e^r\,dr = \frac{\pi}{2}\int_0^1 r\,(e^r)'\,dr$$

$$= \frac{\pi}{2}\left[\left[re^r\right]_0^1 - \int_0^1 e^r\,dr\right]$$

$$= \frac{\pi}{2}\left[e - \left[e^r\right]_0^1\right] = \frac{\pi}{2}\left[e-(e-1)\right]$$

$$= \frac{\pi}{2} \qquad \square$$