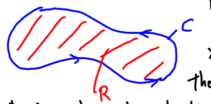
9.12 Green's theorem.



Let C be a simple closed curve in the x,y plane. Then C divides the x,y plane into 2 regions: the "inside of C" and the "outside of C".

Def. The positive direction around C (or positive orientation of C) is the one such that the inside of C is always to the left as the curve is described using this orientation.

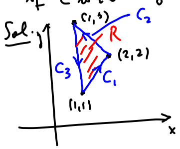
Green's theorem (1st version)

Suppose C is a piecewise smooth simple closed curve bounding a region R in the x,y plane. If $P(x,y)$, $Q(x,y)$, $\frac{\partial P}{\partial y}(x,y)$, $\frac{\partial Q}{\partial x}(x,y)$ are continuous on R, then

$$\boxed{\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA}$$

Ex: Let $P(x,y) = 2(x^2+y^2)$, $Q(x,y) = (x+y)^2$.
Let R be the region bounded by the triangle with vertices at $(1,1)$, $(2,2)$ and $(1,3)$.
Compute $\oint_C P\,dx + Q\,dy$ directly and using Green's theorem if C is the triangle oriented positively.

Sol.



On C_1: let $(x(t), y(t)) = (t,t)$, $1 \le t \le 2$
$x'(t) = y'(t) = 1$

$$\int_{C_1} P\,dx + Q\,dy = \int_1^2 2(2t^2)(1) + (4t^2)(1)\,dt$$
$$= \int_1^2 8t^2\,dt = \left[\frac{8t^3}{3}\right]_1^2 = \frac{56}{3}$$

On C_2: $(x(t), y(t)) = (1-t)(2,2) + t(1,3)$, $0 \le t \le 1$
$= (2-t, 2+t)$ $\quad x'(t) = -1,\ y'(t) = 1$

$$\int_{C_2} P\,dx + Q\,dy = \int_0^1 2((2-t)^2 + (2+t)^2)(-1) + 4^2(1)\,dt$$
$$= \int_0^1 -16 - 4t^2 + 16\,dt = \int_0^1 -4t^2\,dt = \left[-4\frac{t^3}{3}\right]_0^1 = -\frac{4}{3}$$

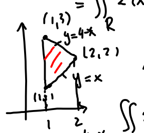
On C_3: $(x(t), y(t)) = (1-t)(1,3) + t(1,1) = (1, 3-2t)$,
$x'(t) = 0,\ y'(t) = -2 \quad 0 \le t \le 1.$

$$\int_{C_3} P\,dx + Q\,dy = \int_0^1 (\dots)(0) + (4-2t)^2(-2)\,dt$$
$$= \left[\frac{(4-2t)^3}{3}\right]_0^1 = \frac{8}{3} - \frac{64}{3} = -\frac{56}{3}$$

$$\oint_C P\,dx + Q\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} \dots = \frac{56}{3} - \frac{4}{3} - \frac{56}{3} = -\frac{4}{3}$$

Using Green's theorem,

$$\int_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_R 2(x+y) - 4y\,dA$$
$$= \iint_R 2(x-y)\,dA$$



R is a region of type I:
$R = \{(x,y),\ 1 \le x \le 2,\ x \le y \le 4-x\}$.

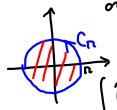
$$\iint_R 2(x-y)\,dA = \int_1^2 \left(\int_x^{4-x} 2(x-y)\,dy\right) dx$$
$$= \int_1^2 \left[-(x-y)^2\right]_{y=x}^{y=4-x} = \int_1^2 -(2x-4)^2\,dx = \left[-\frac{(2x-4)^3}{6}\right]_1^2 = -\frac{8}{6} = -\frac{4}{3}$$

Rem: Let $P(x,y) = \dfrac{-y}{x^2+y^2}$, $Q(x,y) = \dfrac{x}{x^2+y^2}$

$$\frac{\partial Q}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial P}{\partial y} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Let C_r be circle centered at $(0,0)$ with radius $r>0$, oriented positively.



On C_r: let $x(t) = r\cos t$, $y(t) = r\sin t$, $0 \le t \le 2\pi$

$x'(t) = -r\sin t$, $y'(t) = r\cos t$

$$\int_C P\,dx + Q\,dy = \int_0^{2\pi} -\frac{r\sin t}{r^2}(-r\sin t) + \frac{r\cos t}{r^2}\, r\cos t\,dt$$

$$= \int_0^{2\pi} 1\,dt = 2\pi, \quad !$$

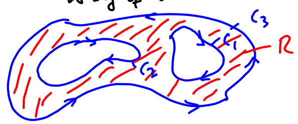
but $\displaystyle\iint_R \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{0}\,dA = 0$

Reason it does not work: both $P(x,y)$ and $Q(x,y)$ are not defined at $(0,0)$!

($P, Q, \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ all need to defined and continuous on all of R)

Regions with holes.

Let R be a bounded region whose boundary consists of finitely many simple closed curves $C_1, C_2, \dots, C_m$ oriented positively (i.e. R should always be on the left as any of these curves is described)

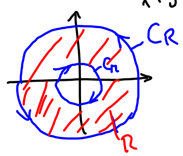


Green's theorem (2d version)

If $P, Q, \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ are all continuous on R, then

$$\boxed{\oint_{C_1} + \oint_{C_2} \cdots + \oint_{C_m} P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA}$$

Ex: $P(x,y) = -\dfrac{y}{x^2+y^2}$, $Q(x,y) = \dfrac{x}{x^2+y^2}$, $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$



$$\oint_{C_R} P\,dx + Q\,dy = 2\pi$$

$$\oint_{C_r} P\,dx + Q\,dy = -2\pi$$

$$0 = \iint_R \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{0} = \oint_{C_r} + \int_{C_R} P\,dx + Q\,dy$$

$$= -2\pi + 2\pi = 0$$