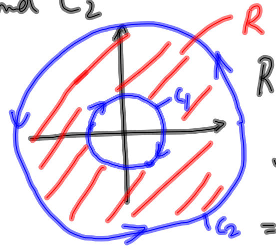
Ex: Use Green's theorem to compute the integral $\oint_C (x^3-y^3)\,dx + (x^3+y^3)\,dy$

where $C = C_1 \cup C_2$ where
C_1 is the circle $x^2+y^2=1$ oriented clockwise
and C_2 " " " $x^2+y^2=9$ " anticlockwise.

Sol.



If we let $R = \{(x,y),\ 1 \le x^2+y^2 \le 9\}$,

$$\oint_C \underbrace{(x^3-y^3)}_{P(x)}dx + \underbrace{(x^3+y^3)}_{Q(x)}\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_R 3(x^2+y^2)\,dA \underset{\text{polar coord}}{=} \int_0^{2\pi}\int_1^3 3r^2\, r\,dr\,d\theta$$

$$= 2\pi \int_1^3 3r^3\,dr = 6\pi\left[\frac{r^4}{4}\right]_1^3 = \frac{3\pi}{2}(81-1) = 120\pi \quad \square$$

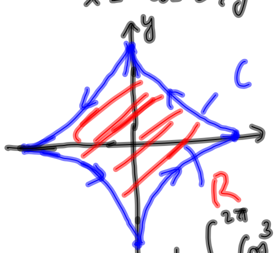
Area of R.

If $P(x,y)=0$ and $Q(x,y)=x$
or if $P(x,y)=-y$ and $Q(x,y)=0$,
then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

$\therefore$ If R is bounded by 1 (a more positively oriented closed curve),

$$\boxed{\text{Area}(R) = \iint_R 1\,dA = \oint_C x\,dy = \oint_C -y\,dx = \frac{1}{2}\int_C x\,dy - y\,dx.}$$

Ex: Use Green's theorem to find the area of the region R bounded by the curve C parametrized by
$x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le 2\pi.$



$x'(t) = -3\cos^2 t \sin t$
$y'(t) = 3\sin^2 t \cos t$

$$A(R) = \iint_R 1\,dA = \frac{1}{2}\int x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} \cos^3 t\,(3\sin^2 t \cos t) - \sin^3 t\,(-3\cos^2 t \sin t)\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t \sin^2 t\,[\underbrace{\cos^2 t + \sin^2 t}_{1}]\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \frac{\sin^2(2t)}{4}\,dt = \frac{3}{8}\int \frac{1-\cos(4t)}{2}\,dt$$

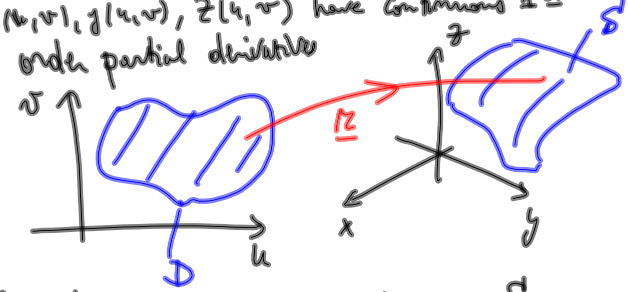
$$= \frac{3}{16}\left[t - \frac{\sin(4t)}{4}\right]_0^{2\pi} = \frac{3\pi}{8}. \quad \square$$

9.13 Surface integrals

Def A <u>parametrized surface</u> is a map $\underline{r} : D \to \mathbb{R}^3$, where D is a region in $\mathbb{R}^2$ and $\underline{r}$ is one-to-one (except possibly at the boundary of D).

$\therefore$ $\underline{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, where $(u,v) \in D$.

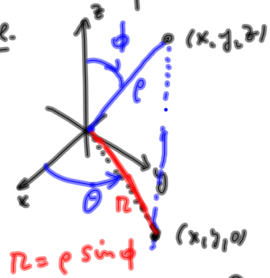
We will assume that the component functions $x(u,v), y(u,v), z(u,v)$ have continuous 1st order partial derivatives



Def: The range of D under $\underline{r}$, i.e. S, is called a surface.

Ex: Find parametric equations for the sphere $x^2+y^2+z^2 = a^2$ $(a>0)$

Sol.



$$\begin{cases} x = r\cos\theta = \rho\cos\theta\sin\phi \\ y = r\sin\theta = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases} \quad \boxed{\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{matrix}}$$

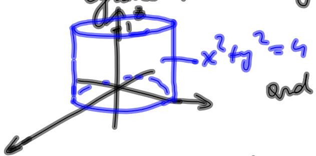
From the equation for the sphere: $\rho^2 = a^2$, i.e. $\rho = a$

So we can use θ, ϕ as parameters

$$\therefore \begin{cases} x = a\cos\theta\sin\phi \\ y = a\sin\theta\sin\phi \\ z = a\cos\phi \end{cases}, \quad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi. \end{matrix}$$

Ex: Find parametric equations for the cylinder $x^2+y^2 = 4$, $0 \le z \le 1$.

Sol.

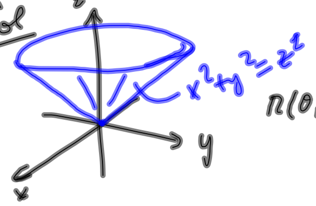


choose z as one parameter and let $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$.

$$\therefore \underline{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle, \quad 0 \le \theta \le 2\pi, \ 0 \le z \le 1.$$

Ex: Find parametric equations for the surface $x^2+y^2 = z^2$, $0 \le z \le 2$

Sol



$$r(\theta, z) = \langle z\cos\theta, z\sin\theta, z \rangle, \quad 0 \le z \le 2, \ 0 \le \theta \le 2\pi \quad \square.$$