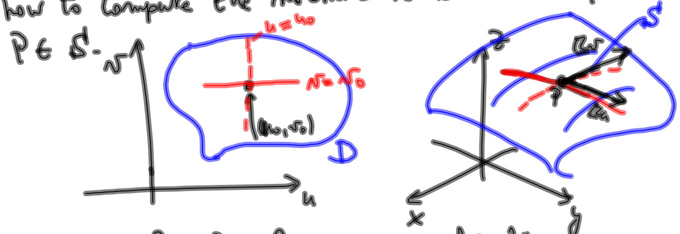
Tangent planes

Given a parametrized surface S parametrized by $\underline{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, $(u,v) \in D$, how to compute the normal to S at some point $P \in S$.



Let $\underline{r}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle$, $\underline{r}_v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \rangle$

Then, both $\underline{r}_u$ and $\underline{r}_v$ are tangent to S at P where $\overrightarrow{OP} = \underline{r}(u_0,v_0)$. A vector normal to S at P is given by:

$$\boxed{\underline{N}(u_0,v_0) = (\underline{r}_u \times \underline{r}_v)_{(u_0,v_0)}}$$

Def: A parametrization is smooth if $\underline{N}(u,v) \neq 0$, for all $(u,v) \in D$.

Ex: Let S be the surface parametrized by $\underline{r}(u,v) = \langle u^2 - v^2, u^2 + v, uv \rangle$, $(u,v) \in \mathbb{R}^2$

- Find an equation for the plane tangent to S at $(0,2,1)$.
- Determine the parameters (u,v) for which the parametrization is smooth.

Sol. We first need to find (u,v) such that

$$\underset{(1)}{u^2 - v^2 = 0}, \quad \underset{(2)}{u^2 + v = 2}, \quad \underset{(3)}{uv = 1}$$

From (1), $v = \pm u$. If $v = -u$, (3) would yield $-u^2 = 1$, which is impossible. So $v = u = \pm 1$

Using (2), $u=-1$ is not possible. $\therefore$ $(u,v) = (1,1)$.

We have $\underline{r}_u = \langle 2u, 2u, v \rangle$

$\underline{r}_v = \langle -2v, 1, u \rangle$

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2u & 2u & v \\ -2v & 1 & u \end{vmatrix} = (2u^2 - v)\,\underline{i} - 2(u^2+v^2)\,\underline{j} + 2u(1+2v)\,\underline{k}$$

$\underline{r}_u \times \underline{r}_v = \underline{N}(u,v)$

$(\underline{r}_u \times \underline{r}_v)_{(1,1)} = \underline{i} - 4\,\underline{j} + 6\,\underline{k}$

The eq. of the plane tangent to S at $(0,2,1)$ is $x - 4(y-2) + 6(z-1) = 0$

or $x - 4y + 6z = -2$

$\underline{N}(u,v) \neq \underline{0}$, so the parametrization is smooth at any $(u,v) \neq (0,0)$

$\Longleftrightarrow (u,v) \neq (0,0)$.

Ex: If S is the graph of function $z = f(x,y)$ where $(x,y) \in D$, we can parametrize S by letting $u = x$, $v = y$:

$$\underline{r}(x,y) = \langle x, y, f(x,y) \rangle, \ (x,y) \in D$$

$$\underline{r}_x = \langle 1, 0, \tfrac{\partial f}{\partial x} \rangle,$$

$$\underline{r}_y = \langle 0, 1, \tfrac{\partial f}{\partial y} \rangle$$

$$\underline{r}_x \times \underline{r}_y = \langle -\tfrac{\partial f}{\partial x}, -\tfrac{\partial f}{\partial y}, 1 \rangle.$$

Definition: (Orientation of a surface)

An orientation of a surface S is a continuous choice of a unit normal vector defined at all points of S.

Rem: If S is parametrized by $\underline{r}(u,v)$ and the parametrization is smooth, a natural orientation for S is obtained by taking the unit normal on S defined by

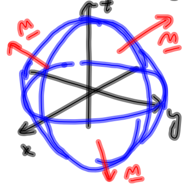
$$\boxed{\underline{n}(u,v) = \frac{\underline{r}_u \times \underline{r}_v}{\| \underline{r}_u \times \underline{r}_v \|}}$$

Ex. Consider the sphere $x^2 + y^2 + z^2 = 1$.

We can choose the orientation of the sphere corresponding to the outward pointing normal.

Since S is the level surface $f(x,y,z) = 1$, where $f(x,y,z) = x^2 + y^2 + z^2$, a normal vector is $\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$ which is outward pointing.

The corresponding unit normal is $\langle x, y, z \rangle$ (on S)



We can also parametrize the sphere using spherical coordinates: $u = \theta$, $v = \phi$

i.e. $\underline{r}(\theta, \phi) = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

$$\underline{r}_\theta = \langle -\sin\theta \sin\phi, \cos\theta \sin\phi, 0 \rangle$$

$$\underline{r}_\phi = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi \rangle$$

$$\underline{r}_\theta \times \underline{r}_\phi = \langle -\cos\theta \sin^2\phi, -\sin\theta \sin^2\phi, -\sin\phi \cos\phi \rangle$$

$$= -\underbrace{\sin\phi}_{\geq 0} \, \underline{r}(\theta, \phi)$$

↳ orientation corresponding to the inward pointing normal