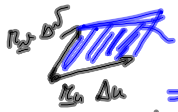
# Surface integrals




Area $= \|r_u \times r_v\|\,\Delta u\,\Delta v$

**Def** Let $S$ be a surface parametrized by $\underline{r}(u,v)$ $((u,v)\in D)$ and let $f: S \to \mathbb{R}$ be a continuous function. Then, we define the surface integral of $f$ on $S$ by the formula:

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\underline{r}(u,v))\,\|r_u \times r_v\|\,du\,dv}$$

**Rem:** (a) This definition is independent of the parametrization chosen for $S$.

(b) If $f \equiv 1$, then 
$$\boxed{\begin{aligned}\iint_S 1\,ds &= \text{Area of } S \\ &= \iint_D \|\underline{r}_u \times r_v\|\,du\,dv\end{aligned}}$$

**Ex:** Evaluate $\iint_S z^2\,ds$ where $S$ is the sphere $x^2+y^2+z^2=1$.

**Sol.** We parametrize $S$ using spherical coordinates:

$\underline{r}(\theta,\phi) = \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle$, $\quad 0\le\theta\le 2\pi$, $0\le\phi\le\pi$.

As before, $r_\theta \times r_\phi = -\sin\phi\ \underline{r}(\theta,\phi)$

$$\|r_\theta \times r_\phi\| = |\sin\phi|\ \underbrace{\|\underline{r}(\theta,\phi)\|}_{1} = \sin\phi$$

$$\therefore \iint_S z^2\,ds = \int_0^{2\pi}\int_0^{\pi} \cos^2\phi\ \sin\phi\ d\phi\ d\theta$$

$$= 2\pi\left[-\frac{\cos^3\phi}{3}\right]_{\phi=0}^{\phi=\pi} = \frac{2\pi}{3}(1+1) = \frac{4\pi}{3} \quad \square$$

**Ex:** Let $S$ be the surface defined by $z = x^2+y$, where $0\le x\le 1$, $-1\le y\le 1$. Find $\iint_S x\,ds$.

**Sol.** Let $\underline{r}(x,y) = \langle x, y, x^2+y\rangle$.

$\underline{r}_x = \langle 1, 0, 2x\rangle$ $\qquad r_x \times \underline{r}_y = \langle -2x, -1, 1\rangle$

$\underline{r}_y = \langle 0, 1, 1\rangle$ $\qquad \|\underline{r}_x \times \underline{r}_y\| = \sqrt{2+4x^2}$

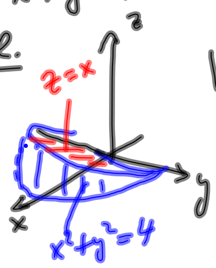
$$\therefore \iint_S x\,ds = \int_0^1\int_{-1}^1 x\sqrt{2+4x^2}\,dy\,dx$$

$$= 2\int_0^1 x\sqrt{2+4x^2}\,dx = 2\left[\frac{(2+4x^2)^{3/2}}{12}\right]_0^1$$

$$= \frac{1}{6}\left[6^{3/2} - 2^{3/2}\right] = \sqrt{6} - \frac{\sqrt{2}}{3} \quad \square$$

**Ex:** Compute the area of the part of the cylinder $x^2+y^2=4$ between the planes $z=0$ and $z=x$, for $x \geq 0$.

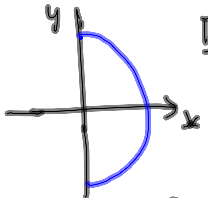
**Sol.**




We let $\underline{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle$,
where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $0 \leq z \leq \underbrace{2\cos\theta}_{x}$

$\underline{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$

$\underline{r}_z = \langle 0, 0, 1 \rangle$




$\underline{r}_\theta \times \underline{r}_z = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$

$\|\underline{r}_\theta \times \underline{r}_z\| = \sqrt{2^2\cos^2\theta + 2^2\sin^2\theta} = 2$

$$\therefore\ A(S) = \iint_S 1\, dS = \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\cos\theta} \|\underline{r}_\theta \times \underline{r}_z\|\, dz\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} 2\, dz\, d\theta = \int_{-\pi/2}^{\pi/2} 4\cos\theta\, d\theta = 4\Big[\sin\theta\Big]_{-\pi/2}^{\pi/2} = 8.$$

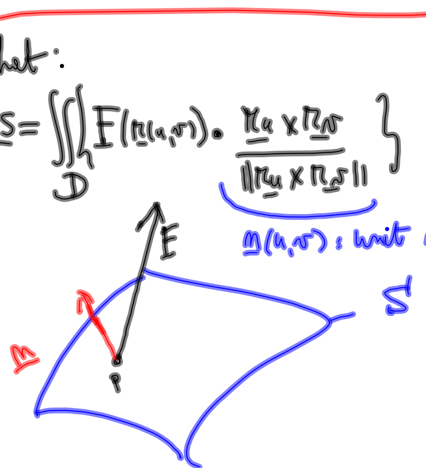
**Def** Surface integrals of vector fields

Let $S$ be parametrized by $\underline{r}(u,v)$ $((u,v) \in D)$ and let $\underline{F}: S \to \mathbb{R}^3$ be a continuous vector field. The surface integral of $\underline{F}$ on $S$ is defined by:

$$\boxed{\iint_S \underline{F} \cdot d\underline{S} = \iint_S \underline{F} \cdot \underline{n}\, dS = \iint_D \underline{F}(\underline{r}(u,v)) \cdot \underline{r}_u \times \underline{r}_v\, du\, dv}$$

Note that:

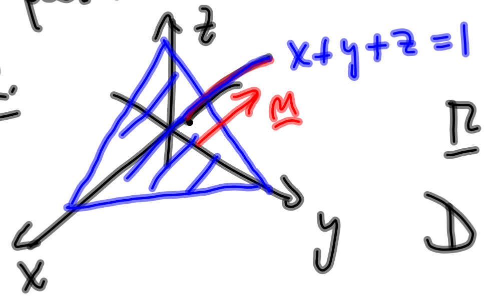
$$\iint_S \underline{F} \cdot d\underline{S} = \iint_D \Big\{ \underline{F}(\underline{r}(u,v)) \cdot \underbrace{\frac{\underline{r}_u \times \underline{r}_v}{\|\underline{r}_u \times \underline{r}_v\|}}_{\underline{n}(u,v):\ \text{unit normal to } S \text{ at } \underline{r}(u,v)} \Big\} \|\underline{r}_u \times \underline{r}_v\|\, du\, dv$$




**Rem:** If we change the orientation of $S$ (i.e. replace $\underline{n}$ by $-\underline{n}$), we change the sign of the integral.

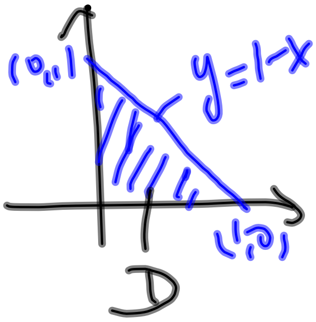
<u>Ex:</u> Compute $\iint_S \underline{F}\cdot d\underline{S}$, where $\underline{F}(x,y,z) = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ and $S$ is the triangle with vertices at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ oriented so that the unit normal makes an angle less than $\frac{\pi}{2}$ with the positive $z$-axis (i.e. its $z$-component is $> 0$).

<u>Sol.</u>




$$\underline{r}(x,y) = \langle x, y, 1-x-y\rangle$$

$$D = \{(x,y),\ 0 \le x \le 1,\ 0 \le y \le 1-x\}$$




$$\underline{r}_x = \langle 1, 0, -1\rangle$$

$$\underline{r}_y = \langle 0, 1, -1\rangle$$

$$\underline{r}_x \times \underline{r}_y = \langle 1, 1, \underbrace{1}_{>0}\rangle$$

$\hookrightarrow$ Correct orientation!

$$\iint_S \underline{F}\cdot d\underline{S} = \iint_D \langle x, y, 1-x-y\rangle \cdot \langle 1, 1, 1\rangle \, dx\, dy$$

$$= \iint_D x + y + 1 - x - y \ \ dx\, dy$$

$$= \iint_D 1 \ dx\, dy = \frac{1}{2}$$

area of D

$\square$