Rem: If the vector field $\underline{F}$ represents the velocity field of a fluid at some particular time t_0, then $\iint_S \underline{F}\cdot d\underline{S}$ represents the net volume of fluid going through the surface at per unit of time (at time t_0). It is also called the flux (or outward flux for a closed surface with outward pointing normal)

Ex: Compute the flux of the vector field $\underline{F}$ given by $\underline{F}(x,y,z) = yz\,\underline{i} + xz\,\underline{j} + xy\,\underline{k}$ across the closed surface consisting of the part of the sphere $x^2+y^2+z^2=1$ above the x,y plane and the part of the x-y plane below the sphere, oriented using the outward pointing normal.

Sol:

S consists of 2 parts S_1 and S_2

S_1: part of the sphere above x,y plane

S_2: part of the plane below the sphere

$$\iint_S \underline{F}\cdot d\underline{S} = \iint_{S_1} \underline{F}\cdot d\underline{S} + \iint_{S_2} \underline{F}\cdot d\underline{S}$$

On S_1: (using spherical coordinates:

$$\underline{r}(\phi,\theta) = \langle \cos\theta \sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle$$

$$\underline{r}_\phi = \langle \cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi\rangle$$

$$\underline{r}_\theta = \langle -\sin\theta\sin\phi,\ \cos\theta\sin\phi,\ 0\rangle$$

$$\underline{r}_\phi \times \underline{r}_\theta = \sin\phi \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle$$

$$= \sin\phi\ \underline{r}(\phi,\theta), \quad 0\le\phi\le \pi/2,\ 0\le\theta\le 2\pi$$

$$\iint_{S_1} \underline{F}\circ \underline{n}\, d\underline{S} =$$

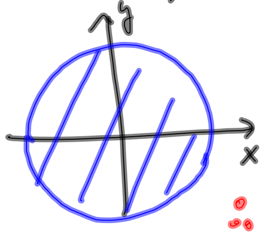
$$\int_0^{2\pi}\int_0^{\pi/2} \langle \sin\phi\cos\phi\sin\theta,\ \sin\phi\cos\phi\cos\theta,\ \sin^2\phi\sin\theta\cos\theta\rangle \circ \sin\phi\langle\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 3\sin^3\phi\cos\phi\sin\theta\cos\theta\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \underbrace{\sin\theta\cos\theta}_{\frac{\sin(2\theta)}{2}}\, d\theta \int_0^{\pi/2} 3\sin^3\phi\cos\phi\, d\phi$$

$$= \underbrace{\left[\frac{-\cos(2\theta)}{4}\right]_0^{2\pi}}_{0} \left[\frac{3}{4}\sin^4\phi\right]_0^{\pi/2} = 0.$$

On S_2: let $\underline{r}(x,y) = \langle x, y, 0\rangle$
for $(x,y) \in D = \{(x,y),\ x^2+y^2 \le 1\}$.



$\underline{r}_x = \langle 1,0,0\rangle$

$\underline{r}_y = \langle 0,1,0\rangle$

$\underline{r}_x \times \underline{r}_y = \langle 0,0,1\rangle \rightarrow$ pointing inward

$\therefore$ We need to multiply $r_x \times r_y$ by -1 to get the correct orientation

$$\iint_{S_2} \underline{F}\cdot d\underline{S} = \iint_D \langle 0,0,xy\rangle \cdot \langle 0,0,-1\rangle \, dA$$

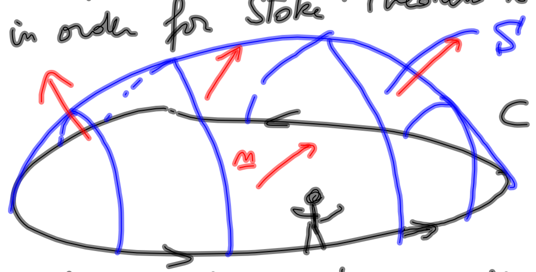
$$= \int_D -xy \, dA = \int_0^{2\pi}\int_0^1 -(r\cos\theta)(r\sin\theta)\, r \, dr\, d\theta$$

$$= -\underbrace{\int_0^{2\pi} \sin\theta\cos\theta\, d\theta}_{0} \int_0^1 r^3\, dr = 0 .$$

$$\iint_S \underline{F}\cdot d\underline{S} = 0$$

9.14 Stokes' theorem.

Suppose that S is an oriented surface which has as boundary a simple closed curve C. The orientation of S and that of C have to "match" in order for Stoke' theorem to hold.



If you walk along C with your head pointing in the direction of the normal $\underline{n}$ on the surface, the surface S should always be on your left

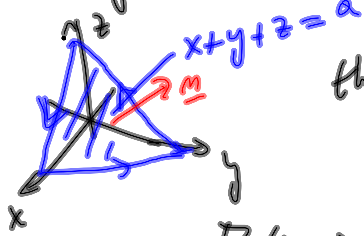
Stokes' theorem.

Let S and C be as above (with compatible orientations) and let $\underline{F}$ be a vector field with continuous 1st order partials on S.

Then
$$\boxed{\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\nabla \times F)\cdot d\underline{S}}$$

Ex: Let S be the triangle with vertices at $(a,0,0)$, $(0,a,0)$, $(0,0,a)$ $(a>0)$ described counterclockwise as viewed from above. Compute $\oint_C \underline{F}\cdot d\underline{r}$ using Stokes' theorem if $\underline{F} = y^2\underline{i} + z^2\underline{j} + x^2\underline{k}$.

Sol.



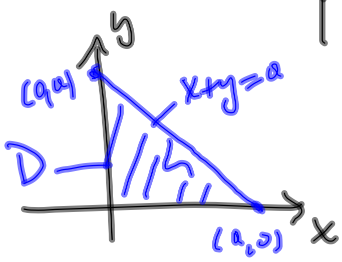
Let S be the part of the plane $x+y+z=a$, bounded by the triangle

$$\underline{r}(x,y) = \langle x, y, a-x-y\rangle$$

$$\underline{r}_x = \langle 1,0,-1\rangle \qquad \underline{r}_x \times \underline{r}_y = \langle 1,1,1\rangle$$
$$\underline{r}_y = \langle 0,1,-1\rangle$$

Correct orientation

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = (-2z)\underline{i} + (-2x)\underline{j} + (-2y)\underline{k}$$



Using Stokes' theorem,

$$\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\nabla\times F)\cdot d\underline{S}$$

$$= \iint_D \langle -2(a-x-y), -2x, -2y\rangle \cdot \langle 1,1,1\rangle \, dx\,dy$$

$$= \iint_D -2a \, dx\,dy$$

$$= -2a \int_0^a \int_0^{a-x} 1 \, dy\, dx = -2a \int_0^a a-x \, dx$$

$$= -2a \left[-\frac{(a-x)^2}{2} \right]_0^a = -2a\,\frac{a^2}{2} = -a^3 \qquad \square$$