Ex: Let C be the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$, with C oriented counterclockwise (as view from above). Evaluate $\int_C \underline{F}\cdot d\underline{r}$ directly and also using Stokes' theorem if $\underline{F}(x,y,z) = -y^3\,\underline{i} + x^3\,\underline{j} - z^3\,\underline{k}$.

Sol:

(i) Directly:

C let $\underline{r}(\theta) = \langle \cos\theta, \sin\theta, 1-\cos\theta-\sin\theta\rangle$, $0 \le \theta \le 2\pi$

$\underline{r}'(\theta) = \langle -\sin\theta, \cos\theta, \sin\theta - \cos\theta\rangle$

$$\oint_C \underline{F}\cdot d\underline{r} = \int_0^{2\pi} \underline{F}(\underline{r}(\theta))\cdot \underline{r}'(\theta)\, d\theta$$

$$= \int_0^{2\pi} \langle -\sin^3\theta, \cos^3\theta, -(1-\cos\theta-\sin\theta)^3\rangle \cdot \langle -\sin\theta, \cos\theta, \sin\theta-\cos\theta\rangle\, d\theta$$

$$= \int_0^{2\pi} \sin^4\theta + \cos^4\theta\, d\theta - \int_0^{2\pi} (1-\cos\theta-\sin\theta)^3(\sin\theta-\cos\theta)\, d\theta = I_1 - I_2.$$

$$I_2 = \left[\frac{(1-\cos\theta-\sin\theta)^4}{4}\right]_0^{2\pi} = 0.$$

$$I_1 = \int_0^{2\pi} \left(\frac{1-\cos(2\theta)}{2}\right)^2 + \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} 2 + 2\cos^2(2\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} 1 + \cos^2(2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{3}{2} + \frac{\cos(4\theta)}{2}\, d\theta = \frac{1}{4}\left[3\theta + \frac{\sin(4\theta)}{4}\right]_0^{2\pi}$$

$$= \frac{3\pi}{2}. \quad \therefore \oint_C \underline{F}\cdot d\underline{r} = I_1 - I_2 = \frac{3\pi}{2}.$$

(ii) Using Stokes' theorem:

Note that C is the boundary of the part of the plane $x+y+z=1$ inside the cylinder $x^2+y^2=1$. Let S be that surface. We can parametrize S by

$\underline{r}(x,y) = \langle x, y, 1-x-y\rangle$, where $(x,y) \in D = \{(x,y),\ x^2+y^2 \le 1\}$.

$\underline{r}_x = \langle 1, 0, -1\rangle$ $\quad$ $\underline{r}_x \times \underline{r}_y = \langle 1, 1, 1\rangle$

$\underline{r}_y = \langle 0, 1, -1\rangle$

this give a compatible orientation for S.

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = 0\,\underline{i} + 0\,\underline{j} + 3(x^2+y^2)\,\underline{k}$$

Using Stokes' theorem,

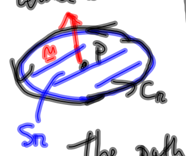
$$\oint_C \underline{F}\cdot d\underline{r} = \iint_S \nabla\times\underline{F}\cdot d\underline{S} = \iint_D (\nabla\times F)(\underline{r}(x,y))\cdot \underline{r}_x\times\underline{r}_y\, dx\,dy$$

$$= \iint_D \langle 0,0,3(x^2+y^2)\rangle\cdot\langle 1,1,1\rangle\, dA = \iint_D 3(x^2+y^2)\, dA$$

$$\underset{\text{polar coord}}{=} \int_0^{2\pi}\int_0^1 3r^2\, r\, dr\, d\theta = 2\pi\left[\frac{3r^4}{4}\right]_0^1 = \frac{3\pi}{2} \quad \square$$

Physical interpretation of the curl

Suppose $\underline{F}$ represents the velocity field of a fluid (at some time $t=t_0$). Let P be a point in the fluid and consider a small disk of radius $r>0$ centered at P with normal vector $\underline{n}$.



Let S_r be the oriented surface and let C_r be the positively oriented boundary circle.

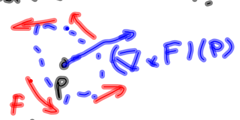
The path integral $\oint_{C_r} \underline{F} \cdot d\underline{r}$ represents the "circulation" of $\underline{F}$ along C_r, i.e. it measures the tendency of the fluid to move around C_r in the positive direction.

By Stokes' Theorem,

$$\oint_{C_r} \underline{F} \cdot d\underline{r} = \iint_{S_r} \nabla \times \underline{F} \cdot d\underline{S} = \iint_{S_r} (\nabla \times \underline{F}) \cdot \underline{n} \, dS$$

$$\approx \underbrace{(\nabla \times \underline{F})(P) \cdot \underline{n}}_{\text{maximized when } \underline{n} \text{ has the same direction as } (\nabla\times \underline{F})(P)} \times \text{Area}(S_r)$$

$\therefore$ the fluid has a tendency to move around an axis through P and direction $(\nabla \times \underline{F})(P)$ in the positive direction.



9.15 Triple integrals

Let $f(x,y,z)$ be a function defined on a closed and bounded region $V \subset \mathbb{R}^3$. We can assume that V is a rectangular box R (by extending $f(x,y,z)$ to be zero outside V). Divide R into smaller boxes R_i, $i=1,\dots,m$ and chose a point P_i in each box R_i.

Form the sum (Riemann sum)

$$I_m = \sum_{i=1}^{m} f(P_i)\,\Delta V_i ,$$

where $\Delta V_i =$ Volume of box R_i

Def. The triple integral of $f(x,y,z)$ on R is

$$\boxed{\iiint_R f(x,y,z)\, dV = \lim_{m\to\infty} I_m}$$

if the limit exists where $\max_i (\Delta V_i) \to 0$ as $m \to \infty$.