Facts:

- If the triple integral of $f(x,y,z)$ on V exists, then $f(x,y,z)$ is "integrable on V".
- If $f(x,y,z)$ is continuous on V, then it is integrable on V.
- $$\boxed{\iiint_V 1 \, dV = \text{Volume of } V.}$$

Fubini's theorem.

If $f(x,y,z)$ is integrable on the rectangular box
$V=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]=\{(x,y,z),\ a_1\le x\le b_1,\ a_2\le y\le b_2,\ a_3\le z\le b_3\}$,

$$\iiint_V f(x,y,z)\,dV = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{a_3}^{b_3} f(x,y,z)\,dz\right)dy\right)dx$$

$$= \int_{a_2}^{b_2}\left(\int_{a_3}^{b_3}\left(\int_{a_1}^{b_1} f(x,y,z)\,dx\right)dz\right)dy$$

$$= \dots \qquad (8 \text{ different ways}).$$

Ex. Let $V=[\frac{1}{6},1]\times[0,\pi]\times[0,2]$.
Compute $\iiint_V xz\sin(xy)\,dV$

Sol. $$\iiint_V xz\sin(xy)\,dV = \int_{\frac{1}{6}}^{1}\int_0^{\pi}\int_0^2 xz\sin(xy)\,dz\,dy\,dx$$

$$= \int_{1/6}^{1}\int_0^{\pi}\left[x\sin(xy)\frac{z^2}{2}\right]_{z=0}^{z=2} dy\,dx = \int_{1/6}^{1}\int_0^{\pi} 2x\sin(xy)\,dy\,dx$$

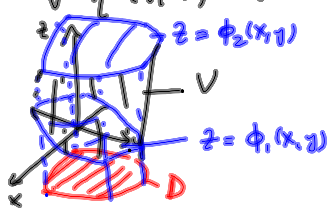
$$= \int_{1/6}^{1}\Big[-2\cos(xy)\Big]_{y=0}^{y=\pi} dx = \int_{1/6}^{1} 2-2\cos(\pi x)\,dx$$

$$= \left[2x - \frac{2\sin(\pi x)}{\pi}\right]_{1/6}^{1} = 2-\left(\frac{1}{3}-\frac{2}{\pi}\frac{1}{2}\right)$$

$$= \frac{5}{3}+\frac{1}{\pi} \qquad \square$$

One can also define type I, II and III regions in 3 dimension. For example, a region of type I is the region between the graphs of 2 functions of x and y defined in some region D in the x,y plane, i.e

$$V=\{(x,y,z),\ (x,y)\in D,\ \phi_1(x,y)\le z\le \phi_2(x,y)\}.$$



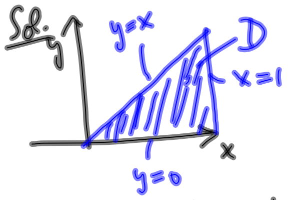
If $f(x,y,z)$ is integrable on V,

$$\boxed{\iiint_V f(x,y,z)\,dV = \iint_D \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Regions of type II and III are dealt with in a similar way.

Ex: Let D be the triangular region in the x,y plane bounded by the lines $y=x$, $x=1$ and $y=0$. Let V be the solid region between the graphs of $z=-y^2$ and $z=x^2$ for $(x,y)\in D$. Evaluate $\iiint_V x+1 \, dV$

Sol.



$D=\{(x,y),\ 0\le x\le 1,\ 0\le y\le x\}$.

$$\iiint_V x+1\,dV=\iint_D\int_{-y^2}^{x^2}(x+1)\,dz\;dA$$

$$=\iint_D (x+1)(x^2+y^2)\,dA=\iint_D (x^3+x^2)+(x+1)y^2\,dA$$

$$=\int_0^1\int_0^x (x^3+x^2)+(x+1)y^2\,dy\,dx$$

$$=\int_0^1\left[(x^3+x^2)y+(x+1)\frac{y^3}{3}\right]_{y=0}^{y=x}dx$$

$$=\int_0^1 x^4+x^3+\frac{x^4+x^3}{3}\,dx=\frac{4}{3}\int_0^1 x^4+x^3\,dx$$

$$=\frac{4}{3}\left[\frac{x^5}{5}+\frac{x^4}{4}\right]_0^1=\frac{4}{3}\,\frac{9}{20}=\frac{3}{5}$$

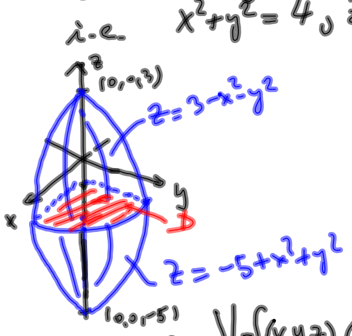
□

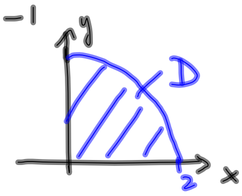
Ex: Let V be the solid region bounded by the paraboloids $z=3-x^2-y^2$ and $z=-5+x^2+y^2$ in the region where $x,y\ge 0$. Compute $\iiint_V y\,dV$.

Sol. The 2 surfaces intersect when

$$z=3-x^2-y^2=-5+x^2+y^2,$$

i.e. $x^2+y^2=4$, $z=-1$





$D=\{(x,y),\ x,y\ge 0,\ x^2+y^2\le 4\}$

$V=\{(x,y,z),\ (x,y)\in D,\ -5+x^2+y^2\le z\le 3-x^2-y^2\}$

$$\therefore \iiint_V y\,dV=\iint_D\left(\int_{-5+x^2+y^2}^{3-x^2-y^2} y\,dz\right)dA=\iint_D y\left(8-2(x^2+y^2)\right)dA$$

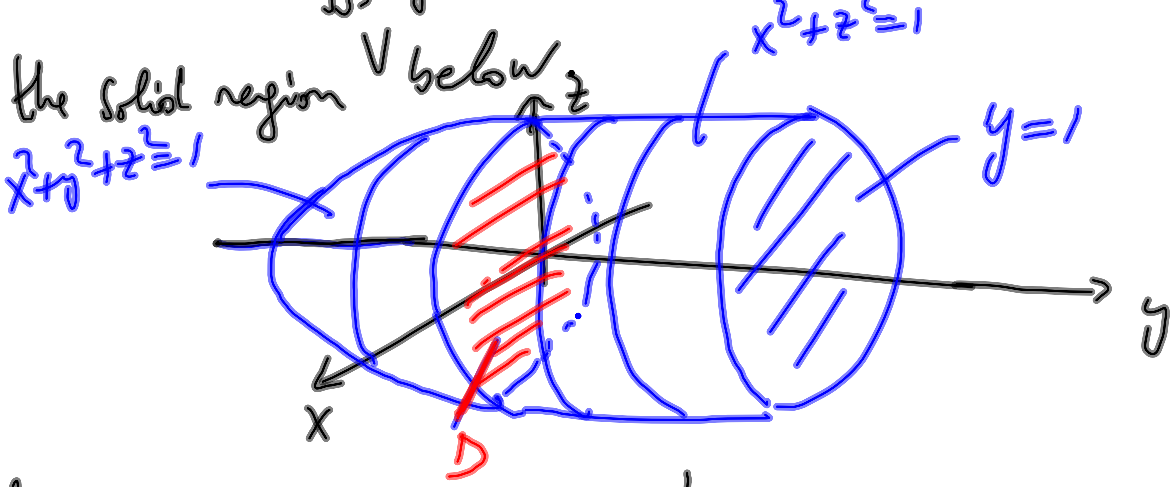
$$=\int_0^{\pi/2}\int_0^2 r\sin\theta\,(8-2r^2)\,r\,dr\,d\theta=\int_0^{\pi/2}\sin\theta\,d\theta\int_0^2 8r^2-2r^4\,dr$$

polar coord

$$=\underbrace{\left[-\cos\theta\right]_0^{\pi/2}}_{1}\left[\frac{8r^3}{3}-\frac{2r^5}{5}\right]_0^2=\frac{64}{3}-\frac{64}{5}=\frac{128}{15}$$

□

Ex: Evaluate $\iiint y\sqrt{1-x^2}\, dV$, where V is the solid region V below



Sol. $D = \{(x,z),\ x^2+z^2 \le 1\}$

$V = \{(x,y,z),\ (x,z)\in D,\ -\sqrt{1-x^2-z^2} \le y \le 1\}.$

$$\iiint_V y\sqrt{1-x^2}\, dV = \iint_D \left[\int_{-\sqrt{1-x^2-z^2}}^{1} y\sqrt{1-x^2}\, dy\right] dx\, dz$$

$$= \cdots = \frac{28}{45}$$