Lecture Notes on Ordinary Differential Equations

Eric T. Sawyer

McMaster University, Hamilton, Ontario
E-mail address: sawyer@mcmaster.ca
URL: http://www.math.mcmaster\~sawyer

Abstract. These lecture notes constitute an elementary introduction to the theory and practice of ordinary differential equations

Contents

Preface

These lecture notes were written during the spring 2012 session at McMaster University in Hamilton, Ontario, and designed for use with Math 2CO3 given that term. I would like to thank my tutorial assistant Chai Molina for several useful comments regarding these notes.

Part 1

Solutions

We begin with some elementary techniques for solving some very special and simple differential equations, and describe the type of equations we will actually try to solve in these notes. We use Picard iterations to prove existence and uniqueness of solutions to initial value problems, and then begin a more systematic investigation of first order and higher order equations, especially linear equations, and derive the method of variation of parameters for solving nonhomogeneous equations. After computing elementary solutions to linear equations with constant coefficients, we turn to the derivation of power series solutions when the coefficients are analytic functions. Then we investigate applications of the Laplace transform to solving linear equations, and end with a more systematic study of first order systems.

CHAPTER 1

Some simple differential equations

Given a function $y(x)$ defined for x in an open interval (a,b) of the real line $\mathbb{R}$, the derivative $y'(x)$ at $x \in (a,b)$ is given by

$$y'(x) \equiv \frac{dy}{dx}(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}, \tag{0.1}$$

provided the limit exists.

0.1. Derivatives and continuity. If the limit in (0.1) *does* exist, then f must be continuous at x. Indeed,

$$\begin{aligned} \lim_{h\to 0} y(x+h) &= \lim_{h\to 0} \left\{ y(x) + \frac{y(x+h) - y(x)}{h} h \right\} \\ &= y(x) + y'(x) \lim_{h\to 0} h = y(x). \end{aligned}$$

The converse however is false.

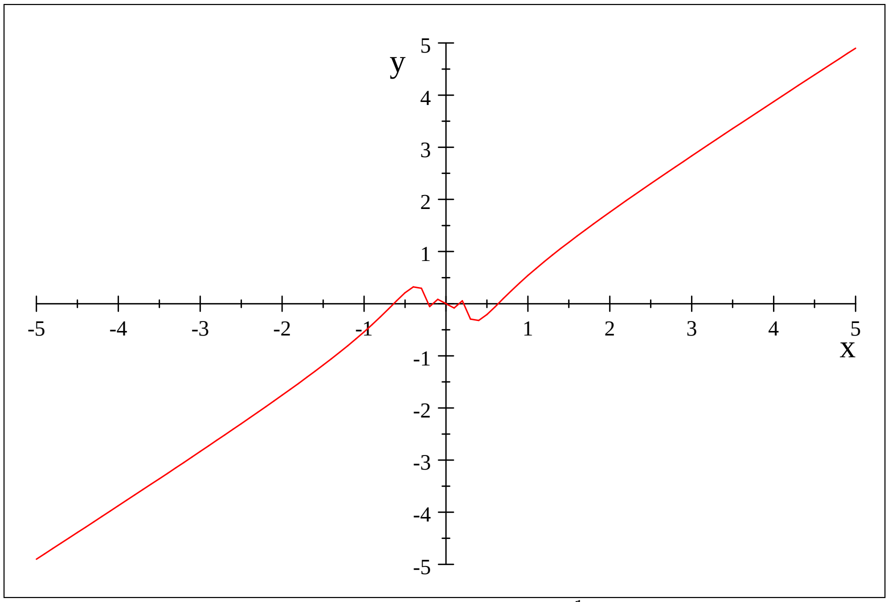
Example 1. *If* $y(x) = \begin{cases} x\cos\frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$*, then for* $x \neq 0$*, the product rule and chain rule from elementary calculus show that* $y'(x)$ *exists, and moreover that* $y'(x)$ *is given by the formula*

$$\begin{aligned} y'(x) &= \left(\frac{d}{dx}x\right)\cos\frac{1}{x} + x\left(\frac{d}{dx}\cos\frac{1}{x}\right) \\ &= \cos\frac{1}{x} - x\sin\frac{1}{x}\left(\frac{d}{dx}\frac{1}{x}\right) \\ &= \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}. \end{aligned}$$

On the other hand at $x = 0$*, we see that* $y'(0)$ *doesn't exist since*

$$\begin{aligned} \frac{y\left(0+\frac{1}{n\pi}\right) - y(0)}{\frac{1}{n\pi}} &= \frac{\frac{1}{n\pi}\cos n\pi - 0}{\frac{1}{n\pi}} \\ &= \cos n\pi = \begin{cases} 1 & \text{if} \quad n \text{ is even} \\ -1 & \text{if} \quad n \text{ is odd} \end{cases}. \end{aligned}$$

Indeed, the difference quotients $\frac{y\left(0+\frac{1}{n\pi}\right)-y(0)}{\frac{1}{n\pi}}$ *are all equal to* 1 *on the sequence* $\left\{\frac{1}{n\pi}\right\}_{n\ even}$ *of numbers that tend to* 0*, while the same difference quotients are all equal to something else, namely* -1*, on the sequence* $\left\{\frac{1}{n\pi}\right\}_{n\ odd}$ *of numbers that also tend to* 0*.*



The graph of $y = x \cos \frac{1}{x}$

0.2. Exponential rates of change. Exponential growth and decay problems from elementary calculus can be modelled by the *differential equation*

$$y' = ky,$$

where k is a real number, referred to as a constant, or constant function. This equation is shorthand for the more explicit formulation

$$y'(x) = ky(x), \qquad x \in (a,b), \tag{0.2}$$

and we say that a function $y(x)$ defined for $x \in (a,b)$ is a *solution* to this equation if $y'(x)$ exists for all $x \in (a,b)$ and if $y'(x) = ky(x)$ for all $x \in (a,b)$. Note that the specification of an interval (a,b) is part of the definition of a solution y.

In elementary calculus it is shown that the exponential functions

$$y_C(x) \equiv Ce^{kx}, \qquad x \in (-\infty, \infty),$$

for C a real constant, are solutions to the differential equation (0.2) when restricted to the interval (a,b). In fact they are the *only* solutions since if $y(x)$ is an arbitrary solution to (0.2), and if we define $C = y(0)$, then assuming $C \neq 0$ (what happens if $C = 0$?) we have

$$\frac{d}{dx}\frac{y(x)}{y_C(x)} = \frac{y_C(x)\, y'(x) - y(x)\, y'_C(x)}{y_C(x)^2} = \frac{Ce^{kx} ky(x) - y(x)\, Cke^{kx}}{y_C(x)^2} = 0.$$

This shows that $\frac{y(x)}{y_C(x)}$ is constant, hence equals the constant $\frac{y(0)}{y_C(0)} = 1$, and we conclude that $y = y_C$.

Example 2. Newton's Law of Cooling *says that a body cools down or warms up at a rate proportional to the difference between the ambient temperature and the temperature of the body. This means that if we define* $T(t)$ *to be the temperature* *of the body at time t, and if the ambient temperature is $\mathcal{T}$, then there is a constant of proportionality k so that*

$$\frac{d}{dt}\left(\mathcal{T}-T\left(t\right)\right)=k\left(\mathcal{T}-T\left(t\right)\right),\qquad t\in\left(-\infty,\infty\right).$$

The solutions are thus given by

$$\mathcal{T}-T\left(t\right)=Ce^{kt},\qquad t\in\left(-\infty,\infty\right),$$

for a real constant C.
For instance, if a body is discovered in a snowbank at 6 am and the temperature has held steady at $-5°\boldsymbol{C}$ overnight, the medical examiner can determine the approximate time of death by taking two readings of the temperature $T\left(t\right)$ of the body (where t is hours since midnight), say at $6:15$ am when $T\left(6.25\right)=13°\boldsymbol{C}$ and again at $6:30$ am when $T\left(6.5\right)=11°\boldsymbol{C}$. Then the two equations

$$\begin{aligned}-5-13 &= \mathcal{T}-T\left(6.25\right)=Ce^{k6.25},\\ -5-11 &= \mathcal{T}-T\left(6.5\right)=Ce^{k6.5},\end{aligned}$$

determine the constant of proportionality k to satisfy

$$e^{k\frac{1}{4}}=\frac{Ce^{k6.5}}{Ce^{k6.25}}=\frac{-5-11}{-5-13}=\frac{8}{9},$$

and the real constant C to satisfy

$$2^5\cdot3^2=\left(-5-13\right)\left(-5-11\right)=C^2e^{k12.75}$$

Assuming the temperature of the body was $37°\boldsymbol{C}$ at the time of death t_{od}, we calculate that

$$\begin{aligned}-5-37 &= \mathcal{T}-T\left(t_{od}\right)=Ce^{kt_{od}}=-\sqrt{\frac{2^5\cdot3^2}{e^{k12.75}}}\left(e^{k\frac{1}{4}}\right)^{4t_{od}}\\ &= -\sqrt{\frac{2^5\cdot3^2}{\left(\frac{8}{9}\right)^{4\cdot(12.75)}}}\left(\frac{8}{9}\right)^{4t_{od}}=-\frac{12\sqrt{2}}{\left(\frac{8}{9}\right)^{25.5}}\left(\frac{8}{9}\right)^{4t_{od}},\end{aligned}$$

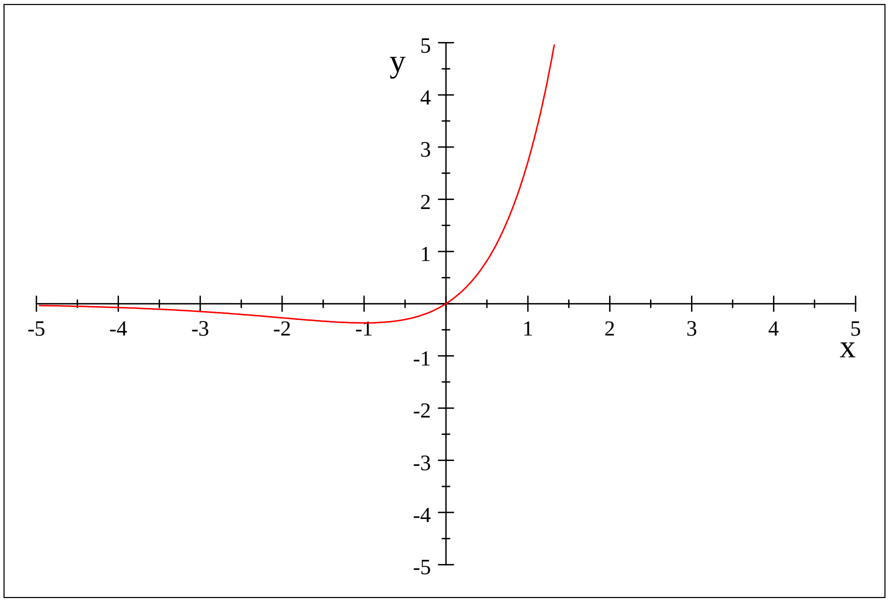
and hence

$$t_{od}=\frac{\ln\frac{42\left(\frac{8}{9}\right)^{25.5}}{12\sqrt{2}}}{4\ln\frac{8}{9}}\approx4.4516,$$

so the time of death was approximately $4:27$ am.

0.3. Classification of differential equations. The function $y=xe^x$ satisfies $y'=\left(1+x\right)e^x$ and so also the equations

$$\begin{aligned}y' &= e^{-x}\frac{y^2}{x}+\frac{y}{x},\\ y'' &= 2y'-y,\\ y\left(\ln\frac{y'}{1+x}\right) &= y'-\frac{y}{x}.\end{aligned}$$



The graph of $y = xe^x$

- The first equation is an example of a *nonlinear* differential equation since the right hand side of $y' = e^{-x}\frac{y^2}{x} + \frac{y}{x}$ is *not* a linear function of the variable y; to be linear it would have to be of the form $f(x)\,y + g(x)$, but there is y^2 in the formula.
- The second equation is an example of a *second order* differential equation since a derivative of order two appears in it. More generally, the *order* of an equation is defined to be the largest order of a derivative appearing in it.
- The third equation involves a *composition* $y \circ \ln \frac{y'}{1+x}$ of the unknown function and a function of its derivative, and is therefore considered 'out of bounds' for us - such equations will not be considered at all in these notes. The equation $y \ln \frac{y'}{1+x} = y' - \frac{y}{x}$ is of course acceptable - it is a first order nonlinear equation.

1. Explicit and implicit families of solutions on intervals

We start with a metaphor. The problem of 'solving' for the solutions x to a polynomial equation

$$P(x) \equiv x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0,$$

begins with solving the 'pure' case $x^n + a_0 = 0$, in which we get $x = \sqrt[n]{-a_0}$ by taking n^{th} roots. This is called solution by radicals, and motivated the classical attempt to solve all polynomial equations by radicals, using clever tricks and substitutions. For example, the general quadratic equation

$$x^2 + a_1 x + a_0 = 0,$$

can be solved by the trick of completing the square

$$x^2 + a_1 x + a_0 = \left(x + \frac{a_1}{2}\right)^2 - \left(\frac{a_1}{2}\right)^2 + a_0,$$

and then taking square roots to get the quadratic formula,

$$x = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}.$$

Cubic equations were solved in a similar spirit by del Ferro (1515) and quartic equations by Ferrari (1545). It was Abel who finally showed that this is impossible for quintic equations, and Galois who gave an alternate proof using one of the most beautiful arguments ever constructed in mathematics.

The problem of 'solving' for the solutions y to a differential equation

$$y^{(n)} = f\left(x, y, y', ..., y^{(n-1)}\right),$$

also begins with solving the 'pure' case $y^{(n)} = f(x)$, in which we get $y = \int ... \int f(x)\, dx$, the n^{th} antiderivative of f. One now hopes to solve more general equations by antidifferentiations, using clever tricks and substitutions. We will begin with some special cases of the first order equation $y' = f(x, y)$.

Consider now the differential equation $\frac{dy}{dx} = y^2 + 1$, i.e.

$$y'(x) = y(x)^2 + 1. \tag{1.1}$$

We can compute the solutions $y(x)$ to (1.1) explicitly by antidifferentiation using the following trick. Since $\frac{d}{dt} \tan^{-1}(t) = \frac{1}{t^2+1}$, we have

$$1 = \frac{y'(x)}{y(x)^2 + 1} = \frac{d}{dx} \tan^{-1}(y(x)),$$

and then integrating from 0 to x gives

$$\begin{aligned} x &= \tan^{-1}(y(x)) - \tan^{-1}(y(0)), \\ y(x) &= \tan\left(x + \tan^{-1}(y(0))\right), \qquad x \geq 0. \end{aligned} \tag{1.2}$$

But the tangent function blows up at $\pm\frac{\pi}{2}$, and so we have blowup when x approaches $x_\infty = \frac{\pi}{2} - \tan^{-1}(y(0))$ for any of the infinitely many values of $\tan^{-1}(y(0))$ (which differ from each other by an integer multiple of π).

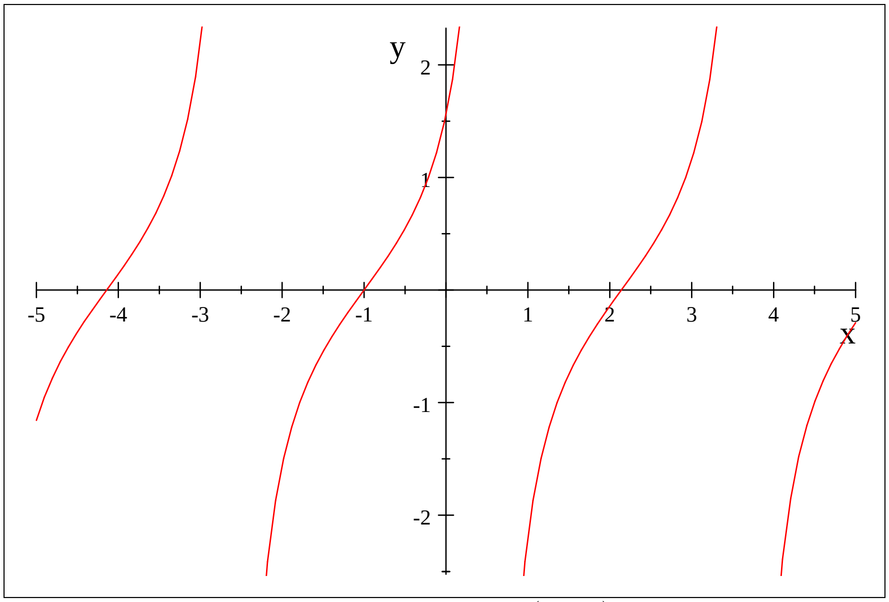
1.1. Families of solutions. The constant $y(0)$ in (1.2) can be taken to be any real number. It is often labelled c and we say that the *general solution* of (1.1) is given by

$$y(x) = \tan\left(x + \tan^{-1}(c)\right), \qquad c \in \mathbb{R}.$$

Here $\tan^{-1}(c)$ is a multi-valued function whose values are all the angles whose tangent is c. By a general solution we mean a family of functions that are solutions, and moreover includes *all* solutions. Since the constants $\tan^{-1}(c)$ cover all of the real numbers for $c \in \mathbb{R}$, we can replace $\tan^{-1}(c)$ by C and write the general solution as

$$y_C(x) = \tan(x + C), \qquad C \in \mathbb{R}.$$

This general solution is called a *one-parameter family* $\{y_c(x)\}_{c\in\mathbb{R}}$ of solutions. However, we must be careful about the domain of definition of solutions.



The graph of $y = \tan(x+1)$

1.2. Intervals of definition of a solution. The solutions $y_C(x)$ are not defined at the singular points $x_\infty = \frac{\pi}{2} - \tan^{-1}(c)$ if $C \in \tan^{-1}(c)$. Now $\tan^{-1}(0) = \{n\pi\}_{n\in\mathbb{Z}}$ and so when $c = 0$ the singular points are

$$x_\infty = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2} - n\pi, \qquad n \in \mathbb{Z},$$

and the *maximal* intervals of definition of the solution $y_0(x) = \tan(x)$ are the open intervals $\{(n\pi,(n+1)\pi)\}_{n\in\mathbb{Z}}$, i.e.

$$..., (-\pi, 0), (0, \pi), (\pi, 2\pi), ...$$

Now we consider the somewhat more challenging differential equation $\frac{dy}{dx} = -\sin(x)\, y^2 + y$, i.e.

$$y'(x) = -\sin(x)\, y(x)^2 + y(x)\,. \tag{1.3}$$

If we make the substitution $v = \frac{1}{y}$ then the equation for the new unknown function $v(x)$ is

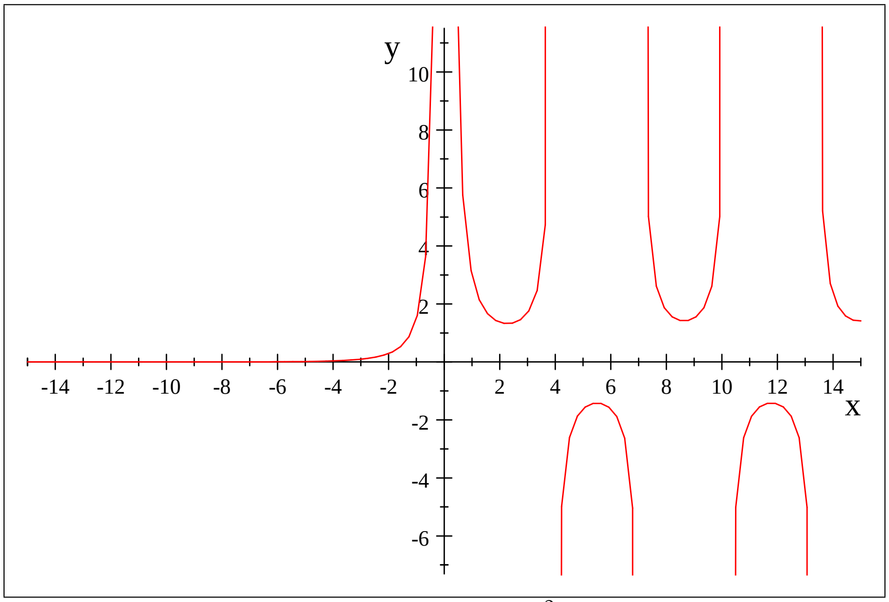
$$\begin{aligned} v'(x) &= \frac{d}{dx}\frac{1}{y(x)} = -\frac{1}{y(x)^2} y'(x) = \sin(x) - \frac{1}{y(x)} = \sin(x) - v(x)\,; \\ v' + v &= \sin x. \end{aligned}$$

This is now a *linear equation* and we can integrate the left hand side if we first multiply by the *integrating factor* $\mu(x) = e^x$:

$$\begin{aligned} \frac{d}{dx}\left(e^x v(x)\right) &= e^x v(x) + e^x v'(x) = e^x (v' + v)(x) = e^x \sin x; \\ e^x v(x) &= \int e^x \sin x = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + \frac{1}{2} C; \\ v(x) &= \frac{1}{2}\left(\sin x - \cos x + Ce^{-x}\right). \end{aligned}$$

Thus a one parameter family of solutions to (1.3) is given by

$$y_C(x) = \frac{1}{v(x)} = \frac{2}{\sin x - \cos x + Ce^{-x}}, \qquad C \in \mathbb{R}. \tag{1.4}$$



The graph of $y = \frac{2}{\sin x - \cos x + e^{-x}}$

Here the solution $y_C(x)$ is defined only for x not equal to a root of the equation $\sin x - \cos x + Ce^{-x} = 0$. For x large enough, Ce^{-x} is negligibly small, and these roots are approximately the roots of the equation $\sin x - \cos x = 0$, i.e. $\tan x = 1$, i.e. $\left\{\frac{\pi}{4} + n\pi\right\}_{n \in \mathbb{Z}}$.

Remark 1 (Caveat). *We may have divided by zero in defining our substitution* $v = \frac{1}{y}$*, and if so we may have lost a* singular solution *in the process. That is indeed the case in our one parameter family of solutions to (1.3) above; the function* $y \equiv 0$ *is a solution to (1.3), but is* not *included in the one parameter family (1.4). The general solution of (1.3) is given by* $\{y_C\}_{C \in \mathbb{R}} \cup \{\mathbf{0}\}$*, where* $\mathbf{0}$ *here denotes the function that is identically zero on the real line. The mystery of singular solutions will be cleared up when we consider* initial value problems *in the next chapter.*

1.3. Implicit solutions. Now we change the differential equation (1.1) to $\frac{dy}{dx} = \frac{1+e^x}{y^2+e^y}$, i.e.

$$y'(x) = \frac{1+e^x}{y(x)^2 + e^{y(x)}}. \tag{1.5}$$

Using

$$\frac{d}{dt}\left(\frac{1}{3}y(t)^3 + e^{y(t)}\right) = \left(y(t)^2 + e^{y(t)}\right) y'(t) = 1 + e^t,$$

we see that we must have the identity or relation,

$$\frac{1}{3}y(x)^3 + e^{y(x)} - \frac{1}{3}y(0)^3 - e^{y(0)} = \int_0^x \left(1 + e^t\right) dt = x + e^x - 1.$$

But this time we cannot solve explicitly for the function $y(x)$ as an elementary function of x. Instead we say that the general solution of (1.5) is given *implicitly* by the family of algebraic equations

$$\frac{1}{3}y^3 + e^y = \left(\frac{1}{3}c^3 + e^c - 1\right) + x + e^x, \qquad c \in \mathbb{R}.$$

Since $\frac{1}{3}c^3 + e^c - 1$ covers all real numbers for $c \in \mathbb{R}$, we can write the general solution implicitly as

$$\frac{1}{3}y^3 + e^y = C + x + e^x, \qquad C \in \mathbb{R}.$$

EXAMPLE 3. *The* folia of Descartes *are the members of the one parameter family of functions defined implicitly by*

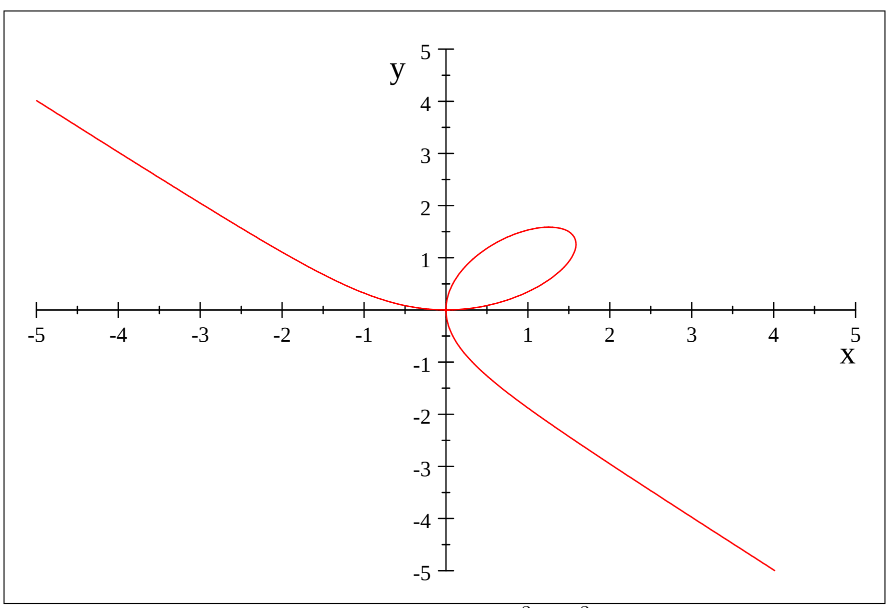
$$x^3 + y^3 = 3Cxy, \qquad C \in \mathbb{R}.$$

To find a differential equation satisfied by this family, solve for the parameter and eliminate it by differentiation,

$$\begin{aligned} 0 &= \frac{d}{dx}(3C) = \frac{d}{dx}\frac{x^3+y^3}{xy} = \frac{d}{dx}\left(\frac{x^2}{y} + \frac{y^2}{x}\right) \\ &= \frac{y2x - x^2y'}{y^2} + \frac{x2yy' - y^2}{x^2} = \left(\frac{2y}{x} - \frac{x^2}{y^2}\right)y' - \left(\frac{y^2}{x^2} - \frac{2x}{y}\right), \end{aligned}$$

to obtain

$$y' = \frac{\frac{y^2}{x^2} - \frac{2x}{y}}{\frac{2y}{x} - \frac{x^2}{y^2}} = \frac{x^2y^2}{x^2y^2}\frac{\frac{y^2}{x^2} - \frac{2x}{y}}{\frac{2y}{x} - \frac{x^2}{y^2}} = \frac{y\left(y^3 - 2x^3\right)}{x\left(2y^3 - x^3\right)}. \tag{1.6}$$



The folium of Descartes $x^3 + y^3 = 3xy$

1.4. Solution techniques. Using the method in the example above, we see that it is quite easy to find a differential equation if we are given a family of prospective solutions. But the reverse problem, that of finding a family of solutions if we are given a differential equation, can be quite daunting. However, all of the equations above were solved using elementary techniques of solution.

(1) Equations (0.2), (1.1) and (1.5) are examples of *separable equations*, i.e. having the form
$$\frac{dy}{dx} = y' = a(x)\, b(y),$$
and were solved by the method of *separation of variables*:
$$\int \frac{dy}{b(y)} = \int a(x)\, dx.$$

(2) Equation (1.3) is an example of a *Bernoulli equation*, i.e. having the form
$$y'(x) = -p(x)\, y(x)^n + q(x)\, y(x),$$
and can be reduced to a *linear equation* by the substitution $v = y^{1-n}$:
$$\begin{aligned} v' &= (1-n)\, y^{-n} y' = (1-n)\, y^{-n} \{-py^n + qy\} \\ &= (n-1)\left\{p - qy^{1-n}\right\} = (n-1)\{p - qv\}; \\ & \quad v' + (n-1)\, qv = (n-1)\, p. \end{aligned}$$

(3) The general linear equation
$$v'(x) + f(x)\, v = g(x),$$
can be integrated after multiplying by the *integrating factor* $\mu(x) = e^{\int^x f}$:
$$\begin{aligned} (\mu(x)\, v(x))' &= \mu v' + \mu' v = \mu\{v' + fv\} = \mu(x)\, g(x); \\ v(x) &= \left(e^{-\int^x f}\right) \int^x \left(e^{\int^t f}\right) g(t)\, dt. \end{aligned}$$

(4) Equation (1.6) is an example of a *homogenous equation*, i.e. having the form
$$\frac{dy}{dx} = y' = f\left(\frac{y}{x}\right),$$
and could have been solved by using the substitution $v = \frac{y}{x}$ to reduce it to a separable equation:
$$f(v) = f\left(\frac{y}{x}\right) = y' = \frac{dy}{dx} = \frac{d}{dx}(xv) = xv' + v; \tag{1.7}$$
$$\text{so } v' = \frac{f(v) - v}{x} \text{ and } \frac{dv}{f(v) - v} = \frac{dx}{x} \text{ give}$$
$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}.$$
For the equation (1.6) we have
$$f(v) = v\frac{v^3 - 2}{2v^3 - 1},$$
and the solution is given implicitly by
$$\ln|x| + c = \int \frac{dx}{x} = \int \frac{dv}{f(v) - v} = \int \frac{1 - 2v^3}{v(1 + v^3)} dv.$$

Maple then gives $\int \frac{1-2v^3}{v(1+v^3)} dv = \ln v - \ln\left(v^3+1\right)$, and we can replace v by $\frac{y}{x}$ in (1.7) and exponentiate to get

$$\begin{aligned}
\pm e^c x &= \frac{\frac{y}{x}}{\frac{y^3}{x^3}+1} = \frac{x^2 y}{y^3+x^3}; \\
x^3 + y^3 &= \pm e^{-c} xy = 3Cxy,
\end{aligned}$$

if we replace $\pm e^{-c}$ with $3C$ (both expressions cover all real numbers).

Here are two *second* order differential equations that are really first order equations disguised by a substitution:

$$y'' = f(x, y') \text{ and } y'' = f(y, y'),$$

where the first equation is missing y on the right, and the second equation is missing x on the right. Both equations are reduced to first order by the substitution $v = y' = \frac{dy}{dx}$, but with *different* independent variables. Namely, the first equation becomes

$$\frac{dv}{dx} = y'' = f(x, y') = f(x, v),$$

which is a first order equation for v as a function of x; while the second equation becomes

$$\frac{dv}{dy} = \frac{dv}{dx}\frac{dx}{dy} = y''\frac{1}{\frac{dy}{dx}} = f(y, y')\frac{1}{y'} = \frac{f(y, v)}{v},$$

which is a first order equation for v as a function of y. For example, to solve

$$y'' = 4y(y')^{\frac{3}{2}},$$

we let $v = y'$ to obtain

$$\begin{aligned}
\frac{dv}{dy} &= \frac{4yv^{\frac{3}{2}}}{v} = 4v^{\frac{1}{2}}y; \\
v^{\frac{1}{2}} &= \int \frac{1}{2}v^{-\frac{1}{2}}dv = \int 2y dy = C_1 + y^2; \\
\frac{dy}{dx} &= v = \left(C_1 + y^2\right)^2; \\
\int \frac{dy}{\left(C_1+y^2\right)^2} &= \int dx = x + C_2,
\end{aligned}$$

which gives x as an antiderivative of $\left(C_1 + y^2\right)^{-2}$, and hence defines an implicit solution. The antiderivative can of course be worked out with partial fractions. The CAS in this editor gives that $\int \frac{dy}{(A+y^2)^2}$ equals $\frac{1}{2A(y^2+A)}$ times

$$\begin{aligned}
&y + \frac{1}{2}A^2\left(\ln\left(y + A^2\sqrt{-\frac{1}{A^3}}\right)\right)\sqrt{-\frac{1}{A^3}} - \frac{1}{2}A^2\left(\ln\left(y - A^2\sqrt{-\frac{1}{A^3}}\right)\right)\sqrt{-\frac{1}{A^3}} \\
&+\frac{1}{2}Ay^2\left(\ln\left(y + A^2\sqrt{-\frac{1}{A^3}}\right)\right)\sqrt{-\frac{1}{A^3}} - \frac{1}{2}Ay^2\left(\ln\left(y - A^2\sqrt{-\frac{1}{A^3}}\right)\right)\sqrt{-\frac{1}{A^3}}.
\end{aligned}$$

Next we investigate an important theoretical point, the existence and uniqueness of *initial value problems*.

CHAPTER 2

Initial value problems

Suppose that $f(x,y)$ is a function of two variables x and y that is defined for (x,y) in a region $\mathcal{R}$ of the plane $\mathbb{R}^2$. We will suppose that $\mathcal{R}$ is *open* in the sense that for any point $P_0 = (x_0, y_0)$ that lies in $\mathcal{R}$, there is some small disk $D(P_0, r)$ of positive radius r centered at P_0 such that $D(P_0, r) \subset \mathcal{R}$. The problem of finding a solution $y(x)$ to the differential equation $y' = f(x,y)$ whose graph passes through a given point $P_0 \in \mathcal{R}$, is called an *initial value problem.* Here is a precise definition of what a solution to an initial value problem is.

Definition 1. *Suppose $\mathcal{R}$ is an open region of the plane $\mathbb{R}^2$, that $f : \mathcal{R} \to \mathbb{R}$ is continuous, and that $P_0 = (x_0, y_0) \in \mathcal{R}$. Then a function $y : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$, where δ is a positive number, is said to be a* solution to the initial value problem

$$\begin{cases} y' &= f(x,y) \\ y(x_0) &= y_0 \end{cases},$$

if these three conditions are met:

(1) *The graph*

$$\mathcal{G} \equiv \left\{(x,y) \in \mathbb{R}^2 : x \in (x_0 - \delta, x_0 + \delta) \text{ and } y = f(x)\right\}$$

of the function $y(x)$ is contained in $\mathcal{R}$,

(2) *The function $y(x)$ is differentiable and satisfies the identity*

$$y'(x) = f(x, y(x)) \qquad \text{for all } x \in (x_0 - \delta, x_0 + \delta),$$

(3) *The function $y(x)$ takes the value y_0 when $x = x_0$, i.e. $y(x_0) = y_0$ or equivalently, $P_0 \in \mathcal{G}$.*

Example 4. *Let $f(x,y) = 3y^{\frac{2}{3}}$ for $(x,y) \in \mathcal{R} \equiv \mathbb{R}^2$ and let $P_0 = (0,0)$. Then f is continuous on $\mathcal{R}$ and $P_0 \in \mathcal{R}$. The associated initial value problem*

$$\begin{cases} y' &= 3y^{\frac{2}{3}} \\ y(0) &= 0 \end{cases}$$

has the trivial solution $y_0(x) \equiv 0$, but also the solution $y_1(x) = x^3$. Thus we see that in general, an initial value problem may have more than one solution.

Example 5. *Let $f(x,y) = y^2 + 1$ for $(x,y) \in \mathcal{R} \equiv \mathbb{R}^2$ and let $P_0 = (0,0)$. The initial value problem*

$$\begin{cases} y' &= y^2 + 1 \\ y(0) &= 0 \end{cases}$$

has the solution $y(x) = \tan x$, but is defined on no larger an interval than $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus we see that in general, a solution to an initial value problem may not be defined on as large an interval as we might expect.

There are two important theorems regarding initial value problems. The first gives conditions under which a solution exists, and the second gives more restrictive conditions under which the solution is unique.

THEOREM 1 (Existence theorem). *Suppose $\mathcal{R}$ is an open region of the plane $\mathbb{R}^2$, that $f:\mathcal{R}\to\mathbb{R}$ is continuous, and that $P_0=(x_0,y_0)\in\mathcal{R}$. Then there* exists *a (possibly very small) positive number δ and a function $y:(x_0-\delta,x_0+\delta)\to\mathbb{R}$ that is a solution to the initial value problem*

$$\left\{\begin{array}{ccc} y' & = & f(x,y) \\ y(x_0) & = & y_0 \end{array}\right. .$$

THEOREM 2 (Uniqueness theorem). *Suppose $\mathcal{R}$ is an open region of the plane $\mathbb{R}^2$, that $f:\mathcal{R}\to\mathbb{R}$ is continuous, and that $P_0=(x_0,y_0)\in\mathcal{R}$. Suppose in addition that $f(x,y)$ satisfies a* Lipschitz *condition in the y variable. This means that there is a positive constant K such that*

$$|f(x,y_1)-f(x,y_2)|\le K|y_1-y_2|, \qquad \textit{for all } (x_1,y_1),(x_2,y_2)\in\mathcal{R}. \tag{0.8}$$

The previous theorem guarantees the existence of a solution to the initial value problem

$$\left\{\begin{array}{ccc} y' & = & f(x,y) \\ y(x_0) & = & y_0 \end{array}\right. ,$$

and this solution is unique *in the sense that any two solutions must agree on their common interval of definition around x_0.*

In applications of the Uniqueness theorem, the Lipschitz hypothesis (0.8) can often be verified using *boundedness* of the partial derivative $\frac{\partial f}{\partial y}$ on the region $\mathcal{R}$. Indeed if

$$\left|\frac{\partial f}{\partial y}(x,y)\right|\le M \text{ for all } (x,y)\in\mathcal{R},$$

then the mean value theorem gives

$$\begin{aligned} |f(x,y_1)-f(x,y_2)| &= \left|\int_{y_2}^{y_1}\frac{\partial f}{\partial y}(x,t)\,dt\right| \\ &\le \left|\int_{y_2}^{y_1}\left|\frac{\partial f}{\partial y}(x,t)\right|dt\right| \le \left|\int_{y_2}^{y_1}M\,dt\right| = M|y_1-y_2|, \end{aligned}$$

so that (0.8) holds with $K=M$.

We can now explain the mystery of the singular solution $\mathbf{0}$ to the Bernoulli equation (1.3).

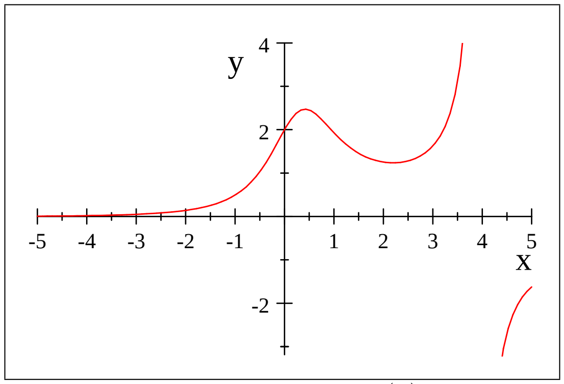
EXAMPLE 6. *The Existence and Uniqueness theorems above apply to the initial value problem*

$$\left\{\begin{array}{ccc} y' & = & -\sin(x)\,y(x)^2+y(x) \\ y(0) & = & y_0 \end{array}\right. , \tag{0.9}$$

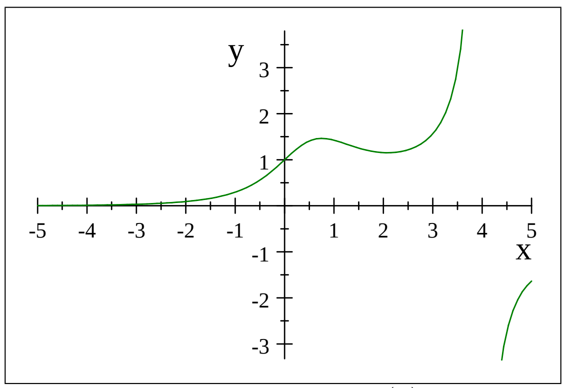
to show that there exists a unique solution (how does Lipschitz apply here?) for every choice of value y_0 for $y(0)$. Now the one parameter family of solutions obtained in (1.4) is

$$y_C(x)=\frac{2}{\sin x-\cos x+Ce^{-x}}, \qquad C\in\mathbb{R},$$

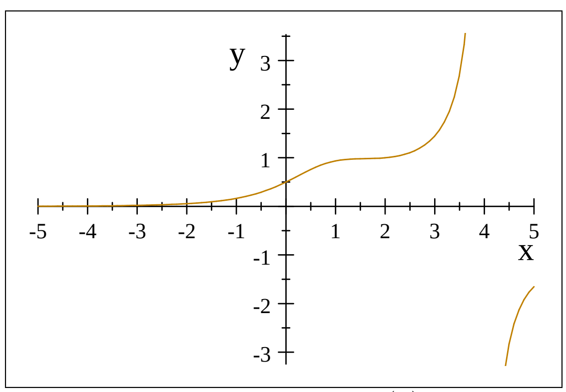
and since $y_0 = y_C(0) = \frac{2}{C-1}$ *for* $C = 1 + \frac{2}{y_0}$*, we see that* $y_{1+\frac{2}{y_0}}(x)$ *is the unique solution to (0.9) when* $y_0 \neq 0$*. When* $y_0 = 0$ *the unique solution is the constant function* $\mathbf{0}$*, which we can interpret as the limit* $\lim_{C\to\infty} y_C(x)$*.*



The graph of $y = y_2(x)$



The graph of $y = y_3(x)$



The graph of $y = y_5(x)$

1. Direction fields

The graphs of solutions $y(x)$ to a differential equation $y' = f(x, y)$ can be visualized, without actually solving the equation, by plotting the associated *direction* *field*

$$\mathbf{v}(x,y) \equiv \frac{(1, f(x,y))}{\|(1, f(x,y))\|} = \left(\frac{1}{\sqrt{1+f(x,y)^2}}, \frac{f(x,y)}{\sqrt{1+f(x,y)^2}} \right).$$

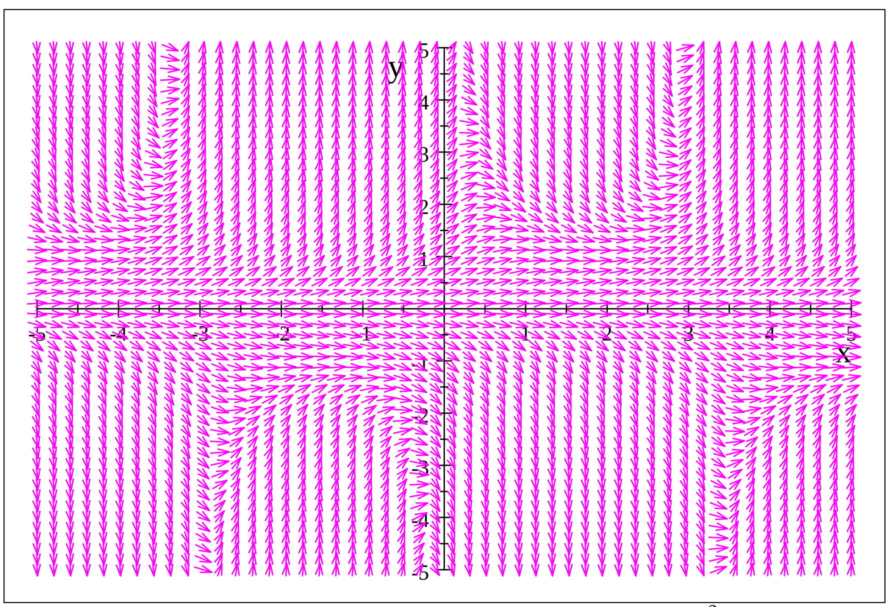
Here the vector $\mathbf{v}(x,y)$ is a unit vector tangent to the graph of the solution $y(x)$ that passes through the point (x,y). This solution $y(x)$ exists and is unique when the hypotheses of the Existence and Uniqueness theorems hold, which is typically the case. This plot provides qualitative information on solutions that is often useful even when explicit solutions are available, since one can 'see' the rough shape of the solution $y(x)$ to an initial value problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases},$$

by starting at the point $P_0 = (x_0, y_0)$ in the plot of the direction field and 'following the arrows'. The direction field for the equation in (0.9) is given by

$$\mathbf{v}(x,y) = \left(\frac{1}{\sqrt{1+(-\sin(x)\,y^2+y)^2}}, \frac{-\sin(x)\,y^2+y}{\sqrt{1+(-\sin(x)\,y^2+y)^2}} \right),$$

and depicted here:



The direction field $\mathbf{v}(x,y)$ for $f(x,y) = -(\sin x)\,y^2 + y$

If you start at the point $(0,2)$ in the plot, and 'follow the arrows', you 'see' the graph of $y = y_2(x)$ displayed above.

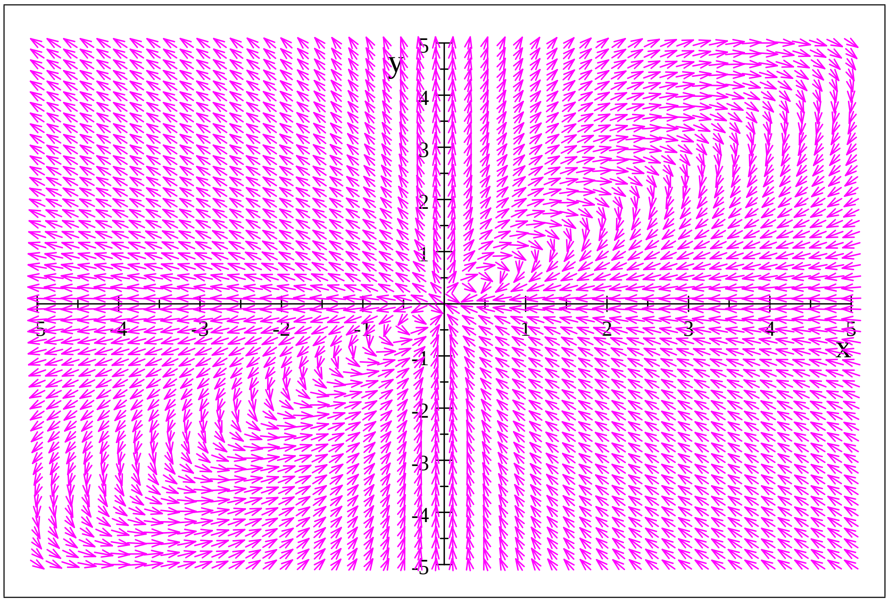
Here is the direction field

$$\left(\frac{x(2y^3-x^3)}{\sqrt{(x(2y^3-x^3))^2+(y(y^3-2x^3))^2}}, \frac{y(y^3-2x^3)}{\sqrt{(x(2y^3-x^3))^2+(y(y^3-2x^3))^2}} \right)$$

for the equation

$$y' = \frac{y\left(y^3 - 2x^3\right)}{x\left(2y^3 - x^3\right)}$$

whose solutions are the folia of Descartes.



2. Picard iterations

We will prove the Existence and Uniqueness theorems assuming the Lipschitz hypothesis (0.8). The proof of the Existence theorem is more difficult without this assumption. There are six basic steps to the existence proof, and the uniqueness is an easy seventh step.

Step 1: We convert the initial value problem

$$\left\{ \begin{array}{ccc} y' & = & f(x,y) \\ y(x_0) & = & y_0 \end{array} \right. . \tag{2.1}$$

into an equivalent *integral equation*:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds. \tag{2.2}$$

Indeed, if y solves the initial value problem (2.1) then part (2) of the fundamental theorem of caluclus shows that

$$y(x) - y_0 = y(x) - y(x_0) = y(s)\mid_{x_0}^{x} = \int_{x_0}^{x} y'(s)\, ds = \int_{x_0}^{x} f(s, y(s))\, ds,$$

which shows that y solves the integral equation (2.2). Conversely, if y solves the integral equation (2.2), then

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(s, y(s))\, ds = y_0 + 0,$$

and part (1) of the fundamental theorem of calculus shows that

$$y'(x) = \frac{d}{dx}\left\{y_0 + \int_{x_0}^{x} f(s, y(s))\, ds\right\} = 0 + f(x, y(x)).$$

Thus y solves the initial value problem (2.1).

The advantage of an integral equation over a differential equation is twofold:

(1) to make sense of each side of the integral equation requires only that y be a *continuous* function, while the differential equation requires that its solutions be *differentiable* functions, a much more restrictive class.

(2) the operation of integration only *improves* functions, i.e. the antiderivative of a continuous function is differentiable, hence continuous; while the operation of differentiation can *worsen* functions, i.e. the derivative of a differentiable function may not be differentiable.

Step 2: We consider Picard's *approximation operator* T which takes a continuous function $\varphi(x)$ to another continuous (actually differentiable!) function $(T\varphi)(x)$ defined by

$$T\varphi(x) = y_0 + \int_{x_0}^{x} f(s, \varphi(s))\, ds.$$

If φ solves the integral equation (2.2) then $\varphi = T\varphi$, and so we can think of the difference

$$\varphi - T\varphi$$

as a function whose 'size' that measures how far φ is from being a solution to (2.2). This is why we call T an approximation operator, and in fact, this suggests the hope that if we start with *any* given continuous function φ, the function $T\varphi$ might be 'closer' to being a solution than φ is. Then $T^2\varphi \equiv T(T\varphi)$, which is T applied to $T\varphi$, might be closer still, and $T^3\varphi$ yet closer.

EXAMPLE 7. *In the case of the initial value problem*

$$\left\{\begin{array}{ccc} y' & = & y \\ y(0) & = & 1 \end{array}\right.,$$

Picard's approximation operator is

$$T\varphi(x) = 1 + \int_0^x \varphi(s)\, ds,$$

and if we choose $\varphi(x) \equiv 1$, *we get*

$$\begin{aligned} T\varphi(x) &= 1 + \int_0^x 1 ds = 1 + x, \\ T^2\varphi(x) &= 1 + \int_0^x (1+s)\, ds = 1 + x + \frac{x^2}{2}, \\ T^3\varphi(x) &= 1 + \int_0^x \left(1 + s + \frac{s^2}{2}\right) ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}, \\ &\vdots \\ T^n\varphi(x) &= 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}, \qquad n \geq 1. \end{aligned}$$

Thus in this case, the Picard approximations $T^n\varphi(x)$ are the Taylor polynomials for e^x, and so converge to $y = e^x$, which is indeed a solution to the given initial value problem.

Recall that in order to define
$$T\varphi = y_0 + \int_{x_0}^{x} f(s, \varphi(s))\, ds,$$
we need that the integrand $f(s, \varphi(s))$ is defined, which in turn requires that $(s, \varphi(s))$ stays in the region $\mathcal{R}$, i.e. that *graph* $\varphi \subset \mathcal{R}$. Similarly, in order to define the iteration
$$T^2\varphi(x) = y_0 + \int_{x_0}^{x} f(s, T\varphi(s))\, ds,$$
we need that the integrand $f(s, T\varphi(s))$ is defined, which in turn requires that $(s, T\varphi(s))$ stays in the region $\mathcal{R}$, i.e. that *graph* $T\varphi \subset \mathcal{R}$.

Step 3: Given positive numbers $\alpha > 0$ and $\beta > 0$ denote by $R_{\alpha,\beta}(x_0, y_0)$ the rectangle
$$R_{\alpha,\beta}(x_0, y_0) \equiv [x_0 - \alpha, x_0 + \alpha] \times [y_0 - \beta, y_0 + \beta].$$
We claim that there are $\alpha > 0$ and $\beta > 0$ (possible quite small) with the two properties
$$R_{\alpha,\beta}(x_0, y_0) \subset \mathcal{R},$$
and
$$\textit{graph } T\varphi \subset R_{\alpha,\beta}(x_0, y_0) \text{ whenever } \textit{graph } \varphi \subset R_{\alpha,\beta}(x_0, y_0).$$
Indeed, if
$$|f(x, y)| \leq M \text{ for all } (x, y) \in \mathcal{R},$$
then we have
$$\begin{aligned} |y_0 - T\varphi(x)| &= \left| \int_{x_0}^{x} f(s, \varphi(s))\, ds \right| \leq \left| \int_{x_0}^{x} |f(s, \varphi(s))|\, ds \right| \\ &\leq M|x - x_0| \leq \alpha \end{aligned}$$
for $(x, y) \in R_{\alpha,\beta}(x_0, y_0)$. So if we choose α small enough to ensure that $M\beta \leq \alpha$, we will have $(x, T\varphi(x)) \in R_{\alpha,\beta}(x_0, y_0)$.

Step 4: Let I denote the interval $[x_0 - \alpha, x_0 + \alpha]$. Consider the vector space of functions
$$\mathcal{C}(I) \equiv \{\varphi : I \to \mathbb{R} : \varphi \text{ is continuous}\}.$$
Define a *distance* function $d(\varphi, \psi)$ that measures the 'distance' between two continuous functions $\varphi, \psi \in \mathcal{C}(I)$ by
$$d(\varphi, \psi) = \max_{x \in I} |\varphi(x) - \psi(x)|.$$
In particular we have
$$|\varphi(x) - \psi(x)| \leq d(\varphi, \psi) \text{ for every } x \in I. \tag{2.3}$$
From Step 3 we conclude that T takes functions φ in $\mathcal{C}(I)$ to functions that are again in $\mathcal{C}(I)$, i.e.
$$T : \mathcal{C}(I) \to \mathcal{C}(I).$$
Such a map T is called a *linear operator* on the vector space of functions $\mathcal{C}(I)$. We think of it as taking a given continuous function φ to a new function $T\varphi$, that is hopefully closer to being a solution to the integral equation (2.2). To get this to actually work we will need to use the Lipschtiz condition (0.8).

Step 5: Start with any fixed $\varphi \in \mathcal{C}(I)$. By Step 4 we can construct the infinite sequence of Picard approximations $\{T^n\varphi\}_{n=0}^{\infty}$ in $\mathcal{C}(I)$, where we set $T^0\varphi = \varphi$ for convenience. Now we estimate the distance between successive approximations $T^n\varphi$ and $T^{n+1}\varphi$. Since

$$\begin{aligned} & T^n\varphi(x) - T^{n+1}\varphi(x) = T\left(T^{n-1}\varphi\right)(x) - T\left(T^n\varphi\right)(x) \\ = & \left(y_0 + \int_{x_0}^{x} f\left(s, T^{n-1}\varphi(s)\right) ds\right) - \left(y_0 + \int_{x_0}^{x} f\left(s, T^n\varphi(s)\right) ds\right) \\ = & \int_{x_0}^{x} \left[f\left(s, T^{n-1}\varphi(s)\right) - f\left(s, T^n\varphi(s)\right)\right] ds, \end{aligned}$$

we have for each $n \geq 1$,

$$\begin{aligned} d\left(T^n\varphi, T^{n+1}\varphi\right) &= \max_{x\in I}\left|T^n\varphi(x) - T^{n+1}\varphi(x)\right| \\ &= \max_{x\in I}\left|\int_{x_0}^{x}\left[f\left(s, T^{n-1}\varphi(s)\right) - f\left(s, T^n\varphi(s)\right)\right] ds\right| \\ &\leq \max_{x\in I}\left|\int_{x_0}^{x} K\left|T^{n-1}\varphi(s) - T^n\varphi(s)\right| ds\right| \\ &\leq \alpha K \max_{s\in I}\left|T^{n-1}\varphi(s) - T^n\varphi(s)\right| \\ &= \alpha K d\left(T^{n-1}\varphi, T^n\varphi\right). \end{aligned}$$

By induction we obtain

$$d\left(T^n\varphi, T^{n+1}\varphi\right) \leq (\alpha K)^n\, d(\varphi, T\varphi), \qquad n \geq 0.$$

Thus by (2.3) we have

$$\left|T^n\varphi(x) - T^{n+1}\varphi(x)\right| \leq d\left(T^n\varphi, T^{n+1}\varphi\right) \leq (\alpha K)^n\, d(\varphi, T\varphi), \qquad x \in I.$$

The infinite series of nonnegative terms,

$$\sum_{n=0}^{\infty}\left|T^n\varphi(x) - T^{n+1}\varphi(x)\right|,$$

will converge provided $\sum_{n=0}^{\infty}(\alpha K)^n < \infty$, i.e. provided $\alpha K < 1$. So let us choose α small enough that $\alpha K \leq \frac{1}{2}$. Then, since absolute convergence implies convergence, we see that the series

$$\sum_{n=0}^{\infty}\left[T^n\varphi(x) - T^{n+1}\varphi(x)\right]$$

must converge to a real number, that we will call $\psi(x)$. But then

$$\begin{aligned} \psi(x) &= \lim_{N\to\infty}\sum_{n=0}^{N}\left[T^n\varphi(x) - T^{n+1}\varphi(x)\right] \\ &= \lim_{N\to\infty}\left\{\left[\varphi(x) - T\varphi(x)\right] + ... + \left[T^N\varphi(x) - T^{N+1}\varphi(x)\right]\right\} \\ &= \lim_{N\to\infty}\left\{\varphi(x) - T^{N+1}\varphi(x)\right\} \end{aligned}$$

shows that

$$\lim_{N\to\infty} T^{N+1}\varphi(x) = \psi(x) - \varphi(x), \qquad x \in I.$$

Step 6: We claim that the limit function

$$y(x) = \psi(x) - \varphi(x), \qquad x \in I,$$

from Step 5 solves the integral equation (2.2), or what is the same thing, that $Ty(x) = y(x)$ for $x \in I$ (one says that y is a *fixed point* of T in this case). Indeed, for $x \in I$,

$$\begin{aligned} y(x) &= \lim_{N\to\infty} T^{N+1}\varphi(x) = \lim_{N\to\infty} T\left(T^N\varphi\right)(x) \\ &= \lim_{N\to\infty}\left\{y_0 + \int_{x_0}^{x} f\left(s, T^N\varphi(s)\right) ds\right\} \\ &= y_0 + \int_{x_0}^{x}\left\{\lim_{N\to\infty} f\left(s, T^N\varphi(s)\right)\right\} ds \\ &= y_0 + \int_{x_0}^{x} f\left(s, \lim_{N\to\infty} T^N\varphi(s)\right) ds \\ &= y_0 + \int_{x_0}^{x} f(s, y(s))\, ds, \end{aligned}$$

where the limit has been taken inside both the integral and the function f. It is a standard theorem in real analysis that limits can be taken inside integrals when the convergence is uniform (which we have here from our definition of distance in $\mathcal{C}(I)$), and of course the limit can be taken inside continuous functions by the definition of continuity.

Step 7: Now we show uniqueness. Suppose that both $y_1(x)$ and $y_2(x)$ are solutions to the initial value problem (2.1). By Step 1 they both solve the integral equation (2.2) and so the difference $y(x) = y_1(x) - y_2(x)$ satisfies

$$\begin{aligned} y(x) &= y_1(x) - y_2(x) \\ &= \left\{y_0 + \int_{x_0}^{x} f(s, y_1(s))\, ds\right\} - \left\{y_0 + \int_{x_0}^{x} f(s, y_2(s))\, ds\right\} \\ &= \int_{x_0}^{x}\left\{f(s, y_1(s)) - f(s, y_2(s))\right\} ds. \end{aligned}$$

From the Lipschitz condition (0.8) we obtain

$$|y(x)| \le \int_{x_0}^{x} K\,|y_1(s) - y_2(s)|\, ds = K\int_{x_0}^{x} |y(s)|\, ds, \tag{2.4}$$

and hence (assuming $y(x)$ is defined for $x_0 \le x \le x_0 + \frac{1}{2K}$)

$$\sup_{x_0\le x\le x_0+\frac{1}{2K}} |y(x)| \le K\frac{1}{2K} \sup_{x_0\le s\le x_0+\frac{1}{2K}} |y(s)| = \frac{1}{2} \sup_{x_0\le x\le x_0+\frac{1}{2K}} |y(x)|.$$

But this implies that $y_1(x) - y_2(x) = y(x) = 0$ for $x_0 \le x \le x_0 + \frac{1}{2K}$. Similarly, $y_1(x) - y_2(x) = 0$ for $x_0 - \frac{1}{2K} \le x \le x_0$.Thus $y_1(x)$ and $y_2(x)$ coincide on the interval $\left[x_0 - \frac{1}{2K}, x_0 + \frac{1}{2K}\right]$, and we can now repeat this argument with the initial point x_0 replaced by the endpoints $x_0 - \frac{1}{2K}$ and $x_0 + \frac{1}{2K}$, and then repeat as often as necessary.

Alternatively, if $y(x_1) \ne 0$ at some point x_1, then in a neighbourhood $\mathcal{N}$ of x_1 we have from (2.4),

$$\left|\frac{d}{dx}\ln|y(x)|\right| = \left|\frac{y'(x)}{y(x)}\right| \le K,$$

which shows that $\ln|y(x)|$ is bounded in the neighbourhood $\mathcal{N}$. But this contradicts the fact that eventually, a large enough neighbourhood of x_1 will have to encounter a point x where $y(x)=0$, this because $y(x_0)=0$.

Problem 1. *Solve the intial value problem*

$$\begin{cases} y' &= 2x(1+y) \\ y(0) &= 0 \end{cases}$$

using Picard approximations $\varphi_n(x) \equiv T^n\varphi_0(x)$ *beginning with* $\varphi_0(x) \equiv 0$.

Solution 1. *We have*

$$T\varphi(x) = \int_0^x 2s(1+\varphi(s))\,ds,$$

and so

$$\begin{aligned}
\varphi_1(x) &= T\varphi_0(x) = \int_0^x 2s(1+0)\,ds = x^2, \\
\varphi_2(x) &= T\varphi_1(x) = \int_0^x 2s\left(1+\left[s^2\right]\right)ds = x^2 + \frac{2}{4}x^4, \\
\varphi_3(x) &= T\varphi_2(x) = \int_0^x 2s\left(1+\left[s^2+\frac{2}{4}s^4\right]\right)ds = x^2 + \frac{2}{4}x^4 + \frac{2\cdot 2}{4\cdot 6}x^6, \\
\varphi_4(x) &= T\varphi_3(x) = \int_0^x 2s\left(1+\left[s^2+\frac{2}{4}s^4+\frac{2\cdot 2}{4\cdot 6}s^6\right]\right)ds \\
&= x^2 + \frac{2}{4}x^4 + \frac{2\cdot 2}{4\cdot 6}x^6 + \frac{2\cdot 2\cdot 2}{4\cdot 6\cdot 8}x^8.
\end{aligned}$$

By induction on n *we obtain*

$$\begin{aligned}
\varphi_n(x) &= x^2 + \frac{2}{4}x^4 + \frac{2\cdot 2}{4\cdot 6}x^6 + \ldots + \frac{2^{n-1}}{4\cdot 6\cdot(2n)}x^{2n} \\
&= x^2 + \frac{1}{2}x^4 + \frac{1}{2\cdot 3}x^6 + \ldots + \frac{1}{n!}x^{2n}, \qquad n \geq 1.
\end{aligned}$$

(Prove the inductive step!) Thus we have

$$\lim_{n\to\infty} T^n\varphi_0(x) = \lim_{n\to\infty}\varphi_n(x) = \sum_{k=1}^{n}\frac{1}{k!}x^{2k} = e^{x^2}-1,$$

and so the unique solution to the initial value problem is $y(x) = e^{x^2}-1$.

CHAPTER 3

Solution methods for first order equations

We can write a first order equation

$$\frac{dy}{dx} = y' = f(x, y)$$

in differential form

$$f(x, y)\, dx - dy = 0.$$

Of course we can then always multiply by any function of two variables $\mu(x, y)$ and get another equation with the same solutions (and maybe more):

$$\mu(x, y) f(x, y)\, dx - \mu(x, y)\, dy = 0.$$

In this chapter we begin by considering the general first order equation in *differential form*

$$M(x, y)\, dx + N(x, y)\, dy = 0. \tag{0.5}$$

1. Exact equations

Consider the differential equation

$$\left(2x + y^2\right) dx + 2xydy = 0,$$

which is neither linear, separable, homogeneous nor Bernoulli. However, a clever observation is that the function $\Phi(x, y) = x^2 + xy^2$ has the properties

$$\frac{\partial}{\partial x}\Phi(x, y) = 2x + y^2 \text{ and } \frac{\partial}{\partial y}\Phi(x, y) = 2xy,$$

which means that our equation can be written as

$$\begin{aligned} 0 &= \left(2x + y^2\right) dx + 2xydy = \frac{\partial}{\partial x}\Phi(x, y) + \frac{\partial}{\partial y}\Phi(x, y)\frac{dy}{dx} \\ &= \frac{d}{dx}\Phi(x, y(x)) = \frac{d}{dx}\left(x^2 + xy^2\right) \end{aligned}$$

by the chain rule if we view $y = y(x)$ as a function of x. Thus we have the family of solutions

$$x^2 + xy^2 = C \text{ or } y = \sqrt{\frac{C}{x} - x}.$$

Note that what made this method work here was the existence of a function $\Phi(x, y)$ with partial derivatives M and N, i.e.$\frac{\partial}{\partial x}\Phi = M$ and $\frac{\partial}{\partial y}\Phi = N$. Such a function can only exist provided $\frac{\partial}{\partial y}M = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\Phi\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\Phi\right) = \frac{\partial}{\partial x}N$ by the equality of mixed second order partial derivatives of Φ.

The above example suggests that we start with a discussion of *exact equations*, which have a very special differential form in which the coefficient functions M and N have their partial derivatives M_y and N_x *equal*:

$$\frac{\partial}{\partial y}M(x,y) = M_y(x,y) = N_x(x,y) = \frac{\partial}{\partial x}N(x,y). \tag{1.1}$$

When this condition holds, we say the equation (0.5) is *exact*. It turns out that in this case there is a function $\Phi(x,y)$ of two variables whose *gradient* $\nabla\Phi = (\Phi_x, \Phi_y)$ equals the vector (M, N):

$$\begin{aligned} \frac{\partial}{\partial x}\Phi(x,y) &= \Phi_x(x,y) = M(x,y); \\ \frac{\partial}{\partial y}\Phi(x,y) &= \Phi_y(x,y) = N(x,y). \end{aligned} \tag{1.2}$$

Note the following calculation, that shows (1.1) and (1.2) are consistent with the *equality of mixed second order partial derivatives*. Indeed, if $\Phi(x,y)$ satisfies the gradient equation (1.2), and if Φ is twice continuously differentiable, then the equality of mixed second order derivatives

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\Phi(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\Phi(x,y),$$

shows that

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\Phi(x,y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\Phi(x,y)\right) = \frac{\partial}{\partial x}N(x,y),$$

which is (1.1). Before proving the existence of such a function $\Phi(x,y)$ for an exact equation, we look at two examples.

EXAMPLE 8. *We consider the equation*

$$\left\{(\cos x)\ln\left(1+x^2\right) + (\sin x + e^y)\frac{2x}{1+x^2}\right\}dx + e^y\ln\left(1+x^2\right)dy = 0,$$

in which

$$\begin{aligned} M &= (\cos x)\ln\left(1+x^2\right) + (\sin x + e^y)\frac{2x}{1+x^2}, \\ N &= e^y\ln\left(1+x^2\right). \end{aligned}$$

Let us now check the exactness condition (1.1) for this pair M and N. We compute

$$M_y = e^y\frac{2x}{1+x^2} \text{ and } N_x = e^y\frac{2x}{1+x^2},$$

so they are indeed equal, and (1.1) holds. If we accept the theorem that says there exists a function $\Phi(x,y)$ satisfying (1.2), then we have the two gradient equations:

$$\begin{aligned} \frac{\partial}{\partial x}\Phi(x,y) &= M = (\cos x)\ln\left(1+x^2\right) + (\sin x + e^y)\frac{2x}{1+x^2}, \\ \frac{\partial}{\partial y}\Phi(x,y) &= N = e^y\ln\left(1+x^2\right). \end{aligned}$$

The second gradient equation looks easier to solve so we start by integrating it with respect to the variable y. We then have

$$\Phi(x,y) = \int e^y\ln\left(1+x^2\right)\ dy = e^y\ln\left(1+x^2\right) + C,$$

but we must be careful here! The constant of integration could be a different constant for each choice of x, in other words, C is really a function of x, which we will call $g(x)$:

$$\Phi(x,y) = e^y \ln\left(1+x^2\right) + g(x).$$

To figure out what this function g is we substitute this answer back into the first gradient equation to get

$$\begin{aligned} & (\cos x)\ln\left(1+x^2\right) + (\sin x + e^y)\frac{2x}{1+x^2} \\ = & \frac{\partial}{\partial x}\left\{e^y \ln\left(1+x^2\right) + g(x)\right\} \\ = & e^y \frac{2x}{1+x^2} + g'(x), \end{aligned}$$

which when solved for $g'(x)$ gives

$$g'(x) = (\cos x)\ln\left(1+x^2\right) + (\sin x)\frac{2x}{1+x^2}.$$

Note how conveniently the variable y dropped out of the right hand side of this equation - otherwise we could not continue to solve it! This is the magic performed by the exactness condition (1.1). Finally we integrate in x to get

$$g(x) = \int \left\{(\cos x)\ln\left(1+x^2\right) + (\sin x)\frac{2x}{1+x^2}\right\} dx = (\sin x)\ln\left(1+x^2\right) + C,$$

where this time C is indeed a constant. Thus we have obtained the implicit solution

$$0 = \Phi(x,y) = e^y \ln\left(1+x^2\right) + g(x) = (\sin x + e^y)\ln\left(1+x^2\right) + C.$$

Example 9. *Consider the equation $y' = \frac{y}{x}$ which we can write as $-ydx + xdy = 0$. This latter equation is not however in exact form since $M_y = -1$ while $N_x = 1$. However if we multiply through by $\frac{1}{x^2+y^2}$ we get an equation*

$$\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0,$$

with

$$M = \frac{-y}{x^2+y^2} \text{ and } N = \frac{x}{x^2+y^2},$$

that is indeed exact since

$$\begin{aligned} \frac{\partial}{\partial y}M &= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-\left(x^2+y^2\right) + y2y}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}, \\ \frac{\partial}{\partial x}N &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right) - x2x}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}. \end{aligned}$$

Then integrating the second gradient equation in y we get

$$\Phi(x,y) = \int N dy = \int \frac{x}{x^2+y^2}dy = \tan^{-1}\frac{y}{x} + g(x),$$

and then substituting this in the first gradient equation we get

$$\frac{-y}{x^2+y^2} = M = \frac{\partial}{\partial x}\Phi(x,y) = \frac{\partial}{\partial x}\left\{\tan^{-1}\frac{y}{x} + g(x)\right\} = \frac{-y}{x^2+y^2} + g'(x),$$

which gives $g'(x) = 0$ and $g(x) = C$. Thus the implicit solution is

$$0 = \tan^{-1}\frac{y}{x} - C,$$

i.e. the rays $\theta = C$ emanating from the origin, and is given expicitly by

$$y = (\tan C)\, x \text{ and } x = 0.$$

Notice however that the function $\Phi(x,y) = \tan^{-1}\frac{y}{x} = \theta$ is not *defined in the open region $\mathcal{R} \equiv \mathbb{R}^2_+ \setminus \{0\}$ where the coefficient functions M and N are defined and infinitely differentiable. The problem with the* global *definition of the function Φ in $\mathcal{R}$ lies in the fact that the coefficient functions M and N have a* singularity *at a point, namely the origin 0, that is* surrounded *by the region $\mathcal{R}$. Of course, this problem with the global definition of Φ does not stop us from solving $\Phi = C$ implicitly for solutions $y(x)$ to the differential equation. But we must be careful in stating our theorem regarding the existence of such functions Φ.*

We say that an open region is simply connected if there are no 'holes' in the region. For example, any disk is simply connected, while the plane minus the origin is not simply connected.

THEOREM 3. *Suppose that $\mathcal{R}$ is an open* simply connected *region in the plane, and that $M(x,y)$ and $N(x,y)$ are continuously differentiable functions in $\mathcal{R}$. Then there is a twice continuously differentiable function $\Phi(x,y)$ satisfying*

$$\left(\frac{\partial}{\partial x}\Phi(x,y), \frac{\partial}{\partial y}\Phi(x,y)\right) = \nabla\Phi(x,y) = (M(x,y), N(x,y)) \tag{1.3}$$

if and only if

$$M_y(x,y) = N_x(x,y). \tag{1.4}$$

PROOF. If (1.3) holds, then the continuity of mixed second order partial derivatives gives $\frac{\partial}{\partial y}M = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\Phi = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\Phi = \frac{\partial}{\partial x}N$, which is (1.4).

Conversely, if (1.4) holds, fix a point $(x_0, y_0) \in \mathcal{R}$ and define a function $\Phi(x_1, y_1)$ as follows:

$$\Phi(x_1, y_1) = \int_{(x_0,y_0)}^{(x_1,y_1)} M(s,t)\, ds + N(s,t)\, dt$$

where the integral $\int_{(x_0,y_0)}^{(x_1,y_1)}$ stands for any *path integral* $\int_{\boldsymbol{\gamma}}$, with $\boldsymbol{\gamma}$ a differentiable path in $\mathcal{R}$ that joins (x_0, y_0) to (x_1, y_1).

The key point here is that the exact condition (1.4) implies that this definition is independent of the defining path we choose to joint (x_0, y_0) to (x_1, y_1)! Indeed, if $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are two such defining paths, and if we assume that the closed path $\boldsymbol{\gamma} - \boldsymbol{\beta}$ surrounds a subregion $\mathcal{D}$ of $\mathcal{R}$, then by Green's theorem applied to the closed path $\boldsymbol{\gamma} - \boldsymbol{\beta}$, we have

$$\begin{aligned}
&\int_{\boldsymbol{\gamma}} \{M(s,t)\, ds + N(s,t)\, dt\} - \int_{\boldsymbol{\beta}} \{M(s,t)\, ds + N(s,t)\, dt\} \\
&\quad = \int_{\boldsymbol{\gamma}-\boldsymbol{\beta}} \{M(s,t)\, ds + N(s,t)\, dt\} = \int_{\partial\mathcal{D}} \{M(s,t)\, ds + N(s,t)\, dt\} \\
&\quad = \int\int_{\mathcal{D}} \left\{-\frac{\partial}{\partial t}M(s,t) + \frac{\partial}{\partial s}N(s,t)\right\} dsdt = \int\int_{\mathcal{D}} 0 dsdt = 0.
\end{aligned}$$

Now it is easy to verify that (1.3) holds. For example, if we choose defining paths $\boldsymbol{\gamma}$ that end in a horizontal segment near the point the (x_1, y_1) then

$$\begin{aligned} \frac{\partial}{\partial x}\Phi(x_1,y_1) &= \lim_{y\to y_1}\frac{\Phi(x_1,y)-\Phi(x_1,y_1)}{y-y_1} \\ &= \lim_{y\to y_1}\frac{\int_{(x_1,y_1)}^{(x_1,y)}\{M(s,t)\,ds+N(s,t)\,dt\}}{y-y_1} \\ &= \lim_{y\to y_1}\frac{\int_{(x_1,y_1)}^{(x_1,y)}M(s,t)\,ds}{y-y_1} = M(x_1,y_1), \end{aligned}$$

since dt vanishes along a horizontal segment, and the average of the continuous function $M(x_1,\cdot)$ on the interval (y_1, y) tends to $M(x_1, y_1)$ as $y \to y_1$. □

2. Integrating factors

Given a first order equation

$$Mdx + Ndy = 0,$$

we say that $\mu = \mu(x, y)$ is an integrating factor for this equation if the equation

$$\mu M dx + \mu N dy = \mu \cdot 0 = 0$$

is exact, i.e.

$$(\mu M)_y = (\mu N)_x.$$

But the product rule gives

$$(\mu M)_y = \mu_y M + \mu M_y \text{ and } (\mu N)_x = \mu_x N + \mu N_x,$$

so that we need

$$\begin{aligned} \mu_y M + \mu M_y &= \mu_x N + \mu N_x; \\ \text{i.e. } M_y - N_x &= -\frac{\mu_y}{\mu}M + \frac{\mu_x}{\mu}N. \end{aligned}$$

In general this partial differential equation is at least as hard to solve for μ as a function of x and y, than our original equation is to solve for y as a function of x. But there are cases in which μ can be easily obtained. Here are two such cases:

Case 1. $\frac{M_y-N_x}{M} = \psi(y)$ *is a function of* y *alone. Then we can solve* $-\frac{\mu_y}{\mu} = \psi(y) = \frac{M_y-N_x}{M}$ *to obtain an integrating factor* $\mu(y)$ *that is also a function of* y *alone.*

Case 2. $\frac{M_y-N_x}{N} = \varphi(x)$ *is a function of* x *alone. Then we can solve* $\frac{\mu_x}{\mu} = \varphi(x) = \frac{M_y-N_x}{N}$ *to obtain an integrating factor* $\mu(x)$ *that is also a function of* x *alone.*

Example 10. *We solve the equation*

$$y(3x+y) + x(x+y)y' = 0.$$

by finding an integrating factor. Now

$$\begin{aligned} M &= y(3x+y) = 3xy + y^2, \\ N &= x(x+y) = x^2 + xy, \\ M_y - N_x &= (3x+2y) - (2x+y) = x + y, \end{aligned}$$

and we note that

$$\frac{M_y - N_x}{N} = \frac{x+y}{x(x+y)} = \frac{1}{x}$$

just happens to be a function of x alone, so that we can apply Case 2 above. First, we solve for the integrating factor $\mu(x)$,

$$\begin{aligned} \frac{\mu_x}{\mu} &= \frac{M_y - N_x}{N} = \varphi(x) = \frac{1}{x}; \\ \ln|\mu| &= \ln|x|; \\ \mu(x) &= x, \end{aligned}$$

and then multiply the equation through by $\mu(x) = \frac{1}{x}$ to get

$$\left(3x^2y + xy^2\right) + \left(x^3 + x^2y\right)y' = 0,$$

which is now an exact equation. To solve this exact equation we compute

$$\begin{aligned} \Phi &= \int M dx = \int \left(3x^2y + xy^2\right) dx = x^3y + \frac{1}{2}x^2y^2 + f(y); \\ x^3 + x^2y &= N = \Phi_y = x^3 + x^2y + f'(y), \end{aligned}$$

which gives $f'(y) = 0$, hence $f(y) = C$. Thus a family of implicit solutions is given by

$$x^3y + \frac{1}{2}x^2y^2 + C = 0.$$

3. Modelling with separable, linear, homogeneous and Bernoulli equations

We consider a variety of 'real life' problems that can be modelled by first order differential equations, and solved using the methods we have developed so far.

PROBLEM 2. *A cannonball of mass m is shot upward, and perpendicular to the earth's surface, with an initial velocity v_0. Assume that the force F of gravity is directed down toward the center of the earth, and has magnitude inversely proportional to the square of the distance from the center of the earth. Moreover, assume that*
(1) *at the surface of the earth F is given by $-mg$ where $g = 9.8\ m/\sec^2$ is the acceleration due to gravity at sea level, and*
(2) *the radius of the earth is $R = 6,371\ km$.*
Assuming there are no friction or other forces acting on the cannonball, find the escape velocity *v_{escape}, the smallest initial velocity v_0 for which the cannonball will* not *return to earth.*

SOLUTION 2. *From **(1)** we have*

$$F = \frac{-k}{(R+h)^2},$$

*where h is the height of the cannonball above the earth's surface, and k is the constant of proportionality. From **(2)** we see that*

$$-mg = \frac{-k}{(R+0)^2}, \quad i.e.\ k = mgR^2,$$

and so we have Newton's law

$$F = \frac{-mgR^2}{(R+h)^2}.$$

Since there are no other forces acting on the cannonball, Newton's second law of motion $F = ma$ then gives

$$\frac{-mgR^2}{(R+h)^2} = F = ma = m\frac{d^2h}{dt^2},$$

which is a second *order differential equation for $h = h(t)$ in terms of time t.*

In order to transform this into a first *order differential equation we rewrite everything in terms of the velocity*

$$v(t) \equiv \frac{d}{dt}h(t),$$

***but viewed as a function of height** $h = h(t)$. Thus we view v as a function of h and eliminate the time variable t in the process. To accomplish this we use*

$$\frac{d}{dt}v(t) = \frac{d^2}{dt^2}h(t) = \frac{-gR^2}{(R+h(t))^2},$$

together with the chain rule $\frac{dv}{dh} = \frac{dv}{dt}\frac{dt}{dh}$ and $1 = \frac{dh}{dt}\frac{dt}{dh}$ to obtain:

$$\frac{dv}{dh} = \frac{dv}{dt}\frac{dt}{dh} = \frac{1}{\frac{dh}{dt}}\frac{dv}{dt} = \frac{1}{v}\frac{dv}{dt} = \frac{1}{v}\frac{-gR^2}{(R+h)^2}.$$

This equation,

$$\frac{dv}{dh} = \frac{1}{v}\frac{-gR^2}{(R+h)^2},$$

is separable and has solution

$$\begin{aligned} \frac{v^2}{2} &= \int v\,dv = -gR^2\int\frac{dh}{(R+h)^2} = gR^2\frac{1}{R+h} + C; \\ v &= \pm\sqrt{g\frac{2R^2}{R+h} + 2C}. \end{aligned}$$

At time $t = 0$ we have $h = 0$ and $v = v_0 > 0$ so that

$$v_0 = \sqrt{2gR + 2C},$$

which gives

$$2C = v_0^2 - 2gR,$$

and thus

$$\begin{aligned} v &= \sqrt{g\frac{2R^2}{R+h} + v_0^2 - 2gR} \\ &= \sqrt{v_0^2 - 2gR\left(1 - \frac{R}{R+h}\right)}. \end{aligned}$$

The cannonball will return to earth if and only if v vanishes at some height $h > 0$ (since then it will reverse direction and start falling back to earth), i.e. if

$$v_0^2 = 2gR\left(1 - \frac{R}{R+h}\right) \text{ for some } h > 0,$$

which means $v_0^2 < 2gR$. Thus with initial velocity $v_0 \geq \sqrt{2gR}$, the cannonball will never return, so $v_{escape} = \sqrt{2gR}$. Using the values given above for g and R we obtain

$$\begin{aligned} v_{escape} &= \sqrt{2\,(9.8)\,\frac{m}{\sec^2}\,(6,371,000)\,m} \\ &\approx 11,175\,\frac{m}{\sec} = 40,230\ km/hr. \end{aligned}$$

PROBLEM 3. *Suppose your nephew has deposited B_0 dollars in a special bank account that pays interest at an annual constant rate r that is compounded every second. Suppose moreover that your nephew actively withdraws and deposits money every second at a constant rate k. Approximately how much money $B(t)$ is in his account after t years? If the initial deposit is \$1,000 at 5% interest, and he withdraws on average \$50 a year, how much is in the bank account after a long time?*

SOLUTION 3. *There are about*

$$n = 365\frac{1}{4}\cdot 24\cdot 60\cdot 60 = 31557600$$

seconds in an average year. If interest is compounded every second, i.e. n times a year, then the value of the bank deposit is increased by a factor of $1+\frac{r}{n}$ each second, and so after t years, the value would be

$$B(t) = B_0\left(1+\frac{r}{n}\right)^{nt} = B_0\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}rt} = B_0\left\{\left(1+\frac{1}{m}\right)^{m}\right\}^{rt}$$

where $m = \frac{n}{r} \ggg 1$ is much larger than 1. It is thus reasonable to approximate the factor $\left(1+\frac{1}{m}\right)^m$ by

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e,$$

and we obtain the approximation

$$B(t) \approx B_0 e^{rt}.$$

Since the function $B_0 e^{rt}$ solves the initial value problem

$$\left\{\begin{array}{rcl} \frac{d}{dt}B &=& rB \\ B(0) &=& B_0 \end{array}\right. ,$$

we are justified in making the approximating assumption that interest is compounded continuously, *i.e.*

$$\frac{d}{dt}B(t) = rB(t)\,, \qquad \text{for all } t>0.$$

If we also approximate the active withdrawals and deposits by a constant rate of change k over time, we obtain the following differential equation for the value $B(t)$ of the bank account after t years:

$$\frac{d}{dt}B(t) = rB(t) + k.$$

This is a linear equation with integrating factor $\mu(t) = e^{-rt}$, *so that*

$$\begin{aligned} \frac{d}{dt}\{\mu(t)B(t)\} &= -re^{-rt}B(t) + e^{-rt}(rB(t)+k) = ke^{-rt}; \\ e^{-rt}B(t) &= \int ke^{-rt}dt = -\frac{k}{r}e^{-rt} + C. \end{aligned}$$

The initial condition $B(0) = B_0$ *gives* $C = B_0 + \frac{k}{r}$ *and we obtain the solution*

$$B(t) = Ce^{rt} - \frac{k}{r} = B_0e^{rt} + \frac{k}{r}\left(e^{rt}-1\right).$$

In the special case $B_0 = 1000$, $r = \frac{1}{20}$ *and* $k = -50$ *we get*

$$B(t) = 1000e^{\frac{t}{20}} - 1000\left(e^{\frac{t}{20}} - 1\right) = 1000,$$

so that the value stays steady at $\$1,000$ *over time.*

PROBLEM 4. *A population of* $S(t)$ *squirrels* increases *over time t from an initial value* S_0, *at a rate* $b(t)S(t)$ *proportional to the number of squirrels* $S(t)$ *at time* t, *and simultaneously* decreases *at a rate* $d(t)\frac{S(t)(S(t)-1)}{2}$ *proportional to the number of pairs of squirrels* $\binom{S(t)}{2}$ *at time* t. *Both the birth and death proportionality functions* $b(t)$ *and* $d(t)$ *are assumed to be positive and to vary periodically over time* t, *e.g. due to seasonal effects. Solve the resulting initial value problem*

$$\left\{\begin{array}{rcl} \frac{d}{dt}S &=& \left(b+\frac{d}{2}\right)S - \frac{d}{2}S^2 \\ S(0) &=& S_0 \end{array}\right. .$$

SOLUTION 4. *The equation is a Bernoulli equation so we make the substitution* $v = S^{1-2} = S^{-1}$ *to get the linear equation*

$$\frac{d}{dt}v = -S^{-2}\frac{d}{dt}S = -S^{-2}\left\{\left(b+\frac{d}{2}\right)S - \frac{d}{2}S^2\right\} = -\left(b+\frac{d}{2}\right)v + \frac{d}{2}.$$

An integrating factor is $\mu(t) = e^{\int_0^t \left(b(x)+\frac{d(x)}{2}\right)dx}$, *and so*

$$\begin{aligned} \frac{d}{dt}\{\mu v\} &= \left(b+\frac{d}{2}\right)\mu v + \mu\left\{-\left(b+\frac{d}{2}\right)v + \frac{d}{2}\right\} = \mu\frac{d}{2}; \\ e^{\int_0^t \left(b(x)+\frac{d(x)}{2}\right)dx}v(t) &= \int_0^t \mu(s)\frac{d(s)}{2}ds; \\ \frac{1}{S(t)} &= v(t) = e^{-\int_0^t \left(b(x)+\frac{d(x)}{2}\right)dx}\left\{\int_0^t \mu(s)\frac{d(s)}{2}ds + C\right\}. \end{aligned}$$

The initial condition $S(0) = S_0$ *gives* $\frac{1}{S_0} = C$ *and so*

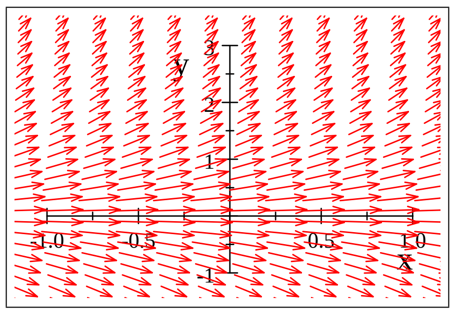
$$\begin{aligned} S(t) &= \frac{1}{e^{-\int_0^t \left(b(x)+\frac{d(x)}{2}\right)dx}\left\{\int_0^t e^{\int_0^s \left(b(x)+\frac{d(x)}{2}\right)dx}\mu(s)\frac{d(s)}{2}ds + C\right\}} \\ &= \frac{1}{\int_0^t e^{\int_t^s \left(b(x)+\frac{d(x)}{2}\right)dx}\frac{d(s)}{2}ds + \frac{1}{S_0}e^{-\int_0^t \left(b(x)+\frac{d(x)}{2}\right)dx}} \\ &= \frac{S_0}{S_0\int_0^t e^{\int_t^s \left(b(x)+\frac{d(x)}{2}\right)dx}\frac{d(s)}{2}ds + e^{-\int_0^t \left(b(x)+\frac{d(x)}{2}\right)dx}}. \end{aligned}$$

4. Euler's numerical tangent line method

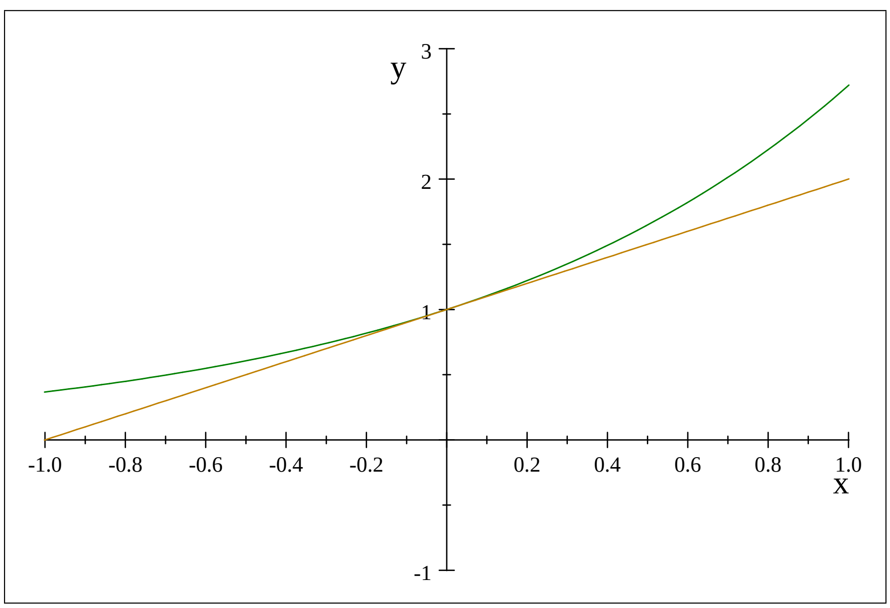
Consider the intial value problem

$$\begin{cases} y' & = & y \\ y(0) & = & 1 \end{cases},$$

whose unique solution we know is $y = e^x$. Suppose that we do not actually know the solution explicitly, and that we wish to compute the numerical value of the solution at $x = 1$, i.e. we want to compute $y(1)$. Or even if we know the answer is $y(1) = e$, we wish to numerically approximate the value $y(1)$. Of course we could use the most crude estimate available, $y(1) \approx y(0) = 1$. However, an inspection of the direction field for this equation



reveals that we can do better by instead approximating $y(1)$ by the value 2 of the tangent line function $x + 1$ at $x = 1$, as pictured below:



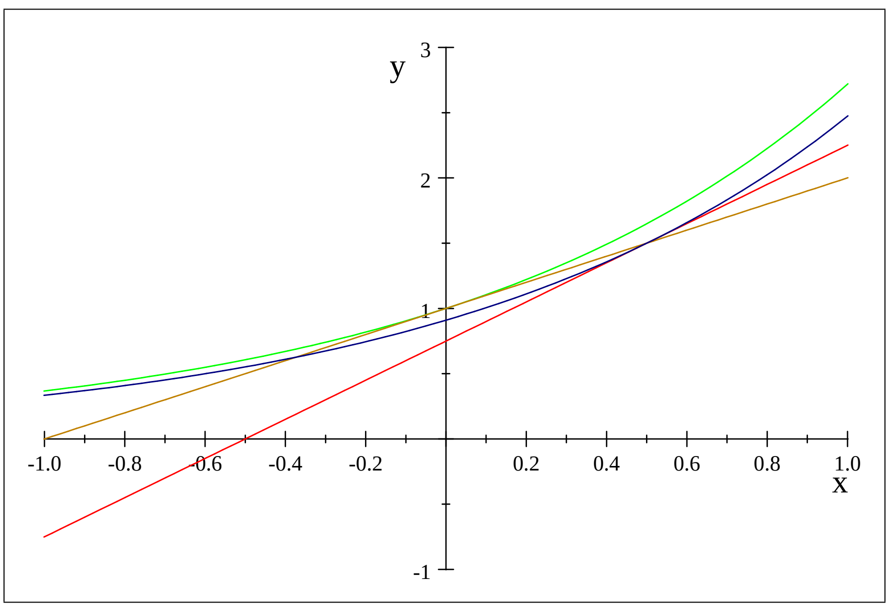
Euler's method with one step

Notice that the divergence between the solution e^x and its tangent line function $x + 1$ increases the further away from the initial point $x = 0$ we go. This suggests that instead, we only go half as far to $x = 0.5$, approximate by the tangent line function $x + 1$ to get $y(0.5) \approx 1 + 0.5 = 1.5$, and then start over with a *new* initial value problem using this approximate value 1.5 as intial condition at the point

$x = 0.5$:

$$\left\{ \begin{array}{ccc} y' & = & y \\ y(0.5) & = & 1.5 \end{array} \right. .$$

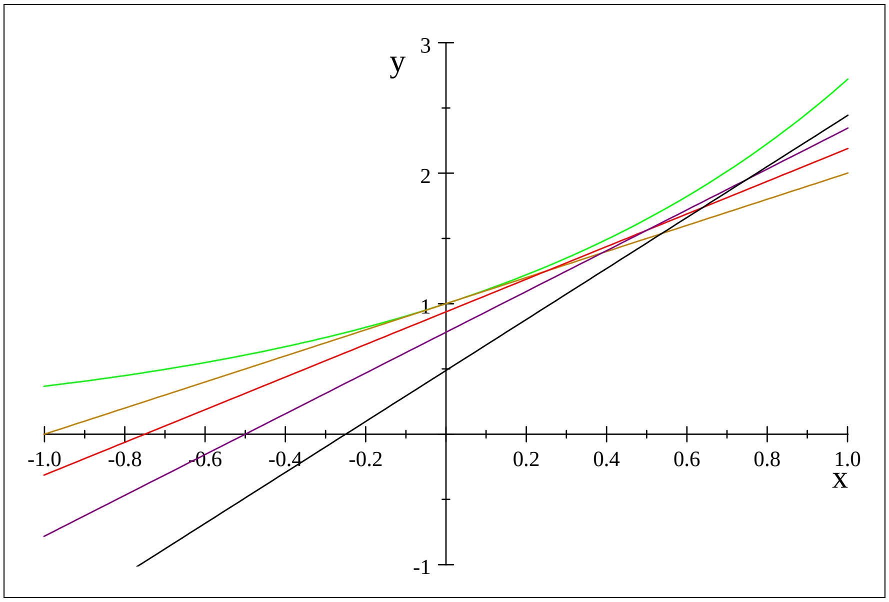
Of course we again know the actual solution to this problem explicitly, namely $y = \frac{3}{2}e^{x-\frac{1}{2}}$, but we do not need this in order to continue! We can just use its tangent line function $\frac{3}{2}x + \frac{3}{4}$ to approximate $y(1)$ as in the picture below. Again, a glance at the direction field indicates that this should indeed give a better approximation than before.



Euler's method with two steps

The graphs of e^x and $\frac{3}{2}e^{x-\frac{1}{2}}$ are the green and blue curves respectively, while their tangent line graphs $x+1$ and $\frac{3}{2}x + \frac{3}{4}$ are the sienna and red lines respectively. The value of the red tangent line at $x = 1$ is $\frac{3}{2} \cdot 1 + \frac{3}{4} = \frac{9}{4} = 2.25$, which is a better approximation to $e = 2.718$ than the result of our first attempt, which gave 2.

We can of course get an even better approximation by applying this procedure *four* times with step size $\frac{1}{4}$ as pictured below. Here the successive tangent line functions are $x + 1$, $\frac{5}{4}x + \frac{15}{16}$, $\frac{25}{16}x + \frac{25}{32}$ and $\frac{125}{64}x + \frac{125}{256}$ (pictured in sienna, red, purple and black), and the final approximation is $\frac{125}{64} \cdot 1 + \frac{125}{256} = \frac{625}{256} = 2.44$, better yet than before.



Euler's method with four steps

Note how our approximations 2, 2.25 and 2.44, obtained from Euler's method with first one, then two and finally four steps, become successively closer to $e = 2.718$. But they don't appear to converge very rapidly! In fact we can dramatically improve the rate of convergence by using a modification of these steps due to Runge and Kutta, and this will be addressed in a later chapter.

4.1. The general setup. Here is the general setup for applying Euler's numerical method to approximate solutions to the initial value problem

$$\begin{cases} y' &= f(x,y) \\ y(x_0) &= y_0 \end{cases} .$$

Pick a (small) positive number h, called the step size, and define points (x_n, y_n) successively in the plane by

$$\begin{aligned} x_1 &= x_0 + h \text{ and } y_1 = y_0 + f(x_0,y_0)\,h, \\ x_2 &= x_1 + h \text{ and } y_2 = y_1 + f(x_1,y_1)\,h, \\ x_3 &= x_2 + h \text{ and } y_3 = y_2 + f(x_2,y_2)\,h, \\ &\vdots \\ x_{n+1} &= x_n + h \text{ and } y_{n+1} = y_n + f(x_n,y_n)\,h, \\ &\vdots \end{aligned}$$

as long as $f(x_n, y_n)$ is defined. Note that we can rewrite the general inductive step as

$$y_{n+1} = y_n + f(x_n, y_n)\,h \ ,$$

where $x_n = x_0 + nh$.

EXAMPLE 11. *We use Euler's method with step size* $h = 0.1$ *to approximate the solution to the initial value problem*

$$\begin{cases} y' &= x\sqrt{y} \\ y(1) &= 4 \end{cases},$$

at the points $x = 1.1, 1.2, 1.3, 1.4$ *and* 1.5. *We are given the data* $f(x,y) = x\sqrt{y}$ *and* $(x_0, y_0) = (1,4)$. *Then the general formulas are*

$$\begin{aligned} x_n &= x_0 + n(0.1), \\ y_{n+1} &= y_n + (x_n)(\sqrt{y_n})(h), \end{aligned}$$

and so we compute

$$\begin{aligned} x_1 &= 1 + 0.1 = 1.1, \\ y_1 &= 4 + (1)\left(\sqrt{4}\right)(0.1) = 4.2, \end{aligned}$$

then

$$\begin{aligned} x_2 &= 1 + 2(0.1) = 1.2, \\ y_2 &= 4.2 + (1.1) \cdot \sqrt{4.2}\,(0.1) = 4.42543, \end{aligned}$$

then

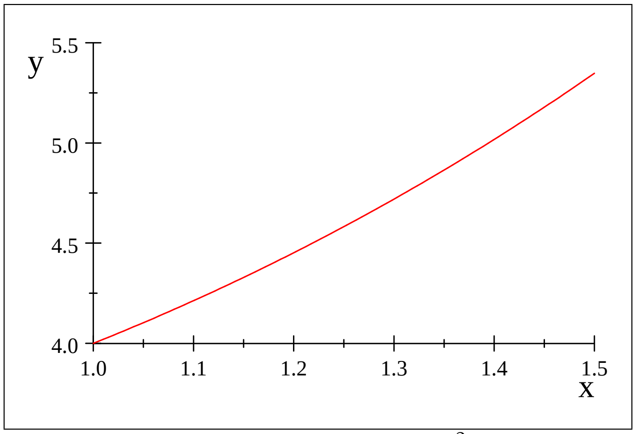
$$\begin{aligned} x_3 &= 1 + 3(0.1) = 1.3, \\ y_3 &= 4.42543 + (1.1) \cdot \sqrt{4.42543}\,(0.1) = 4.45210, \end{aligned}$$

which leads to the following table:

$$\begin{bmatrix} \mathbf{n} & \mathbf{x_n} & \mathbf{y_n} & \mathbf{y(x_n)} \\ 0 & 1 & 4 & 4 \\ 1 & 1.1 & 4.2 & 4.21276 \\ 2 & 1.2 & 4.42543 & 4.45210 \\ 3 & 1.3 & 4.67787 & 4.71976 \\ 4 & 1.4 & 4.95904 & 5.01760 \\ 5 & 1.5 & 5.27081 & 5.34766 \end{bmatrix},$$

where we have included the values $y(x_n)$ *of the exact solution* $y(x) = \left(\frac{x^2+7}{4}\right)^2$, *which is obtained from*

$$2\sqrt{y} = \int \frac{dy}{\sqrt{y}} = \int x\,dx = \frac{x^2}{2} + C \text{ and } 2\sqrt{4} = \frac{1^2}{2} + C.$$



The graph of $y = \left(\frac{x^2+7}{4}\right)^2$.

It is more common to approximate the value of the solution to an initial value problem

$$\begin{cases} y' &= f(x,y) \\ y(x_0) &= y_0 \end{cases},$$

at a fixed point x, and to use a *fixed* number n of steps in the Euler method to get to x, i.e. to use step size

$$h = \frac{x - x_0}{n}.$$

Then $x_n = x_0 + nh = x$ and so we use y_n as our approximation to $y(x_n) = y(x)$.

EXAMPLE 12. *Consider again the initial value problem*

$$\begin{cases} y' &= x\sqrt{y} \\ y(1) &= 4 \end{cases},$$

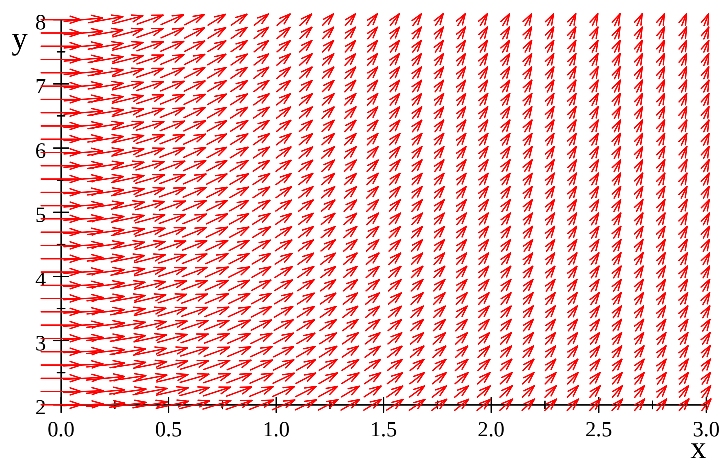
and this time approximate the solution at $x = 1.5$ *using* $n = 10$ *steps. Then* $h = \frac{1.5-1}{10} = 0.05$ *and we get*

$$y_{n+1} = y_n + (1 + (0.05)\,n)\sqrt{y_n}\,(0.05)\,.$$

We will now compute the values y_n *iteratively but rounding these values off to only four decimal places. This leads to the sequence*

$$\begin{aligned}
y_0 &= 4 \\
y_1 &= 4 + (1)\sqrt{4}\,(0.05) = 4.1 \\
y_2 &= 4.1 + (1.05)\sqrt{4.1}\,(0.05) = 4.2063 \\
y_3 &= 4.2063 + (1.1)\sqrt{4.2063}\,(0.05) = 4.3191 \\
y_4 &= 4.3191 + (1.15)\sqrt{4.3191}\,(0.05) = 4.4386 \\
y_5 &= 4.4386 + (1.2)\sqrt{4.4386}\,(0.05) = 4.565 \\
y_6 &= 4.565 + (1.25)\sqrt{4.5654}\,(0.05) = 4.6985 \\
y_7 &= 4.6985 + (1.3)\sqrt{4.6985}\,(0.05) = 4.8394 \\
y_8 &= 4.8394 + (1.35)\sqrt{4.8394}\,(0.05) = 4.9879 \\
y_9 &= 4.9879 + (1.4)\sqrt{4.9879}\,(0.05) = 5.1442 \\
y_{10} &= 5.1442 + (1.45)\sqrt{5.1442}\,(0.05) = 5.3086.
\end{aligned}$$

Note how using ten steps has resulted in an estimate $y_{10} = 5.3086$ that is *closer* to the actual value $y(1.5) = 5.34766$ than the previous estimate 5.27081 using just five steps. Since the solutions to the equation $y' = x\sqrt{y}$ are all convex up in the vicinity of the solution $y(x)$ whose graph passes through $(1,4)$ (as an inspection of the direction field reveals), the Euler approximation to $y(1.5)$ will always be *less* than $y(1.5)$ for any $n \geq 1$, but will increase as n increases.



Direction field of $y' = x\sqrt{y}$.

In general, the error in using Euler's method for small step sizes, is bounded by some constant mulitple C (that depends on the nature of the function $f(x,y)$ near the initial point (x_0, y_0)) times the step size h, i.e. $Error \leq Ch$. However, the error can actually be worse than this because of *roundoff error*, the additional error introduced when rounding off the intermediate values y_1,...,y_9.

CHAPTER 4

Higher order differential equations

In this chapter we consider n^{th} order differential equations of the form

$$y^{(n)} = f\left(x, y, y', ..., y^{(n-1)}\right), \tag{0.1}$$

where $n > 1$ and $y^{(k)}$ denotes the k^{th} derivative of y as a function of x:

$$y^{(k)}(x) = \frac{d^k}{dx^k} y(x) = \overbrace{\frac{d}{dx} \cdots \frac{d}{dx}}^{k\ times} y(x).$$

A solution $y(x)$ to equation (0.1) on an interval I is an n times differentiable function $y : I \to \mathbb{R}$ that satisfies the identity

$$y^{(n)}(x) = f\left(x, y(x), y'(x), ..., y^{(n-1)}(x)\right), \qquad x \in I.$$

Example 13. *The functions* $y_1(x) = \sin x$ *and* $y_2(x) = \cos x$ both *satisfy the second order equation*

$$y'' + y = 0. \tag{0.2}$$

Moreover the collection of functions

$$\{c_1 \sin x + c_2 \cos x\}_{(c_1, c_2) \in \mathbb{R}^2}$$

is a two*-parameter family of solutions to (0.2).*

1. Equivalence with first order systems

It is an extremely useful theoretical observation that the n^{th} order equation (0.1) is equivalent to the following system of n first order equations for the n unknown functions $y_1(x)$, $y_2(x)$,...$y_n(x)$,

$$\left\{ \begin{array}{ccc} y_1' & = & y_2 \\ y_2' & = & y_3 \\ \vdots & \vdots & \vdots \\ y_{n-1}' & = & y_n \\ y_n' & = & f(x, y_1, y_2, ..., y_{n-1}) \end{array} \right., \tag{1.1}$$

in the sense that $y(x)$ solves (0.1) if and only if $y(x) = y_1(x)$ where the functions $\{y_1, y_2, ..., y_n\}$ solve the system (1.1). Indeed, if $y(x)$ solves (0.1), then the set of functions $\{y_1, y_2, ..., y_n\} = \left\{y, y', ..., y^{(n-1)}\right\}$ clearly satisfies the first $n-1$ equations in (1.1) by definition; and the final equation is satisfied because (0.1) gives

$$y_n' = \left(y^{(n-1)}\right)' = y^{(n)}(x) = f\left(x, y, y', ..., y^{(n-1)}\right) = f(x, y_1, y_2, ..., y_{n-1}).$$

Conversely, if $\{y_1, y_2, ..., y_n\}$ satisfies (1.1) and $y = y_1$, then the first $n-1$ equations in (1.1) give

$$y_k = y'_{k-1} = ... = y_1^{(k-1)} = y^{(k-1)}, \qquad 1 \le k \le n,$$

by induction on k; and then the final equation in (1.1) gives

$$y_1^{(n)} = \left(y_1^{(n-1)}\right)' = y'_n = f\left(x, y_1, y_2, ..., y_{n-1}\right) = f\left(x, y, y', ..., y^{(n-1)}\right),$$

which is (0.1).

Of course the system (1.1) is very special in that the first $n-1$ equations are extremely simple. The *general* first order system of n equations in n unknown functions $\{y_1, y_2, ..., y_n\}$ is

$$\left\{\begin{array}{ccc} y'_1 & = & f_1\left(x, y_1, y_2, ..., y_{n-1}\right) \\ y'_2 & = & f_2\left(x, y_1, y_2, ..., y_{n-1}\right) \\ \vdots & \vdots & \vdots \\ y'_{n-1} & = & f_{n-1}\left(x, y_1, y_2, ..., y_{n-1}\right) \\ y'_n & = & f_n\left(x, y_1, y_2, ..., y_{n-1}\right) \end{array}\right., \tag{1.2}$$

where the functions $f_k\left(x, y_1, y_2, ..., y_{n-1}\right)$ are typically arbitrary for $k = 1, 2, ..., n-1, n$. Without further thought, it might appear that the the $n \times n$ first order system (1.2) is simply much *worse* than the n^{th} order equation (0.1).

However, matters appear much brighter if we rewrite the system (1.2) in vector form

$$\mathbf{y}' = \mathbf{f}\left(x, \mathbf{y}\right), \tag{1.3}$$

where we use boldface type to denote n-dimensional vectors,

$$\begin{aligned} \mathbf{y} &= \left(y_1, y_2, ..., y_n\right) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \\ \mathbf{f}\left(x, \mathbf{y}\right) &= \left(f_1\left(x, \mathbf{y}\right), f_2\left(x, \mathbf{y}\right), ..., f_n\left(x, \mathbf{y}\right)\right) = \begin{pmatrix} f_1\left(x, \mathbf{y}\right) \\ f_2\left(x, \mathbf{y}\right) \\ \vdots \\ f_n\left(x, \mathbf{y}\right) \end{pmatrix}, \end{aligned}$$

which we write as either row vectors or column vectors depending on context. In the vector form (1.3), our system looks much more like the first order scalar equation $y' = f\left(x, y\right)$ considered in our Existence and Uniqueness theorems above.

Indeed, if we simply replace numbers by vectors in the appropriate places in the seven step proof of the Existence and Uniqueness theorems above, we obtain Existence and Uniqueness theorems for *systems* of first order equations that read almost exactly the same!

THEOREM 4 (Existence theorem). *Suppose $\mathcal{R}$ is an open region of the Euclidean space $\mathbb{R}^{n+1}$, that $\mathbf{f} : \mathcal{R} \to \mathbb{R}^n$ is continuous, and that $\mathbf{P}_0 = \left(x_0, \mathbf{y}_0\right) \in \mathcal{R}$. Then there* exists *a (possibly very small) positive number δ and a vector function $\mathbf{y} : \left(x_0 - \delta, x_0 + \delta\right) \to \mathbb{R}^n$ that is a solution to the $n \times n$ initial value problem*

$$\left\{\begin{array}{ccc} \mathbf{y}' & = & \mathbf{f}\left(x, \mathbf{y}\right) \\ \mathbf{y}\left(x_0\right) & = & \mathbf{y}_0 \end{array}\right..$$

THEOREM 5 (Uniqueness theorem). *Suppose $\mathcal{R}$ is an open region of the Euclidean space $\mathbb{R}^{n+1}$, that $\mathbf{f} : \mathcal{R} \to \mathbb{R}^n$ is continuous, and that $\mathbf{P}_0 = (x_0, \mathbf{y}_0) \in \mathcal{R}$. Suppose in addition that $\mathbf{f}(x, \mathbf{y})$ satisfies a* Lipschitz *condition in the $\mathbf{y}$ variables. This means that there is a positive constant K such that*

$$\|\mathbf{f}(x, \mathbf{y}_1) - \mathbf{f}(x, \mathbf{y}_2)\| \leq K \|\mathbf{y}_1 - \mathbf{y}_2\|, \qquad \text{for all } (x_1, \mathbf{y}_1), (x_2, \mathbf{y}_2) \in \mathcal{R}.$$

The previous theorem guarantees the existence of a solution to the $n \times n$ initial value problem

$$\left\{ \begin{array}{ccc} \mathbf{y}' & = & \mathbf{f}(x, \mathbf{y}) \\ \mathbf{y}(x_0) & = & \mathbf{y}_0 \end{array} \right. ,$$

and this solution is unique *in the sense that any two solutions must agree on their common interval of definition around x_0.*

In the theorems above, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ and $\mathbf{f}(x, \mathbf{y}) = \begin{pmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \\ \vdots \\ f_n(x, \mathbf{y}) \end{pmatrix}$ are n-dimensional vectors. The length of a vector $\mathbf{y}$ is $\|\mathbf{y}\| = \sqrt{y_1^2 + ... + y_n^2}$ and the notation $\mathbf{y}_1$ and $\mathbf{y}_2$ is used to denote two *different* vectors - the subscripts here do not stand for components of the vectors.

When applied to the n^{th} order equation (0.1), these theorems give existence and uniqueness for an *initial value problem* for (0.1), one that involves specifying the values of the derivatives $y^{(k)}(x_0)$ for $k = 0, 1, 2, ..., n-1$ at an initial point x_0.

THEOREM 6. *Suppose $\mathcal{R}$ is an open region of the Euclidean space $\mathbb{R}^{n+1}$, that $f : \mathcal{R} \to \mathbb{R}$ is continuous, and that $\mathbf{P}_0 = (x_0, \mathbf{y}_0) \in \mathcal{R}$, where $\mathbf{y}_0 = \left(y_0^0, y_0^1, ..., y_0^{n-1}\right)$ is a point in $\mathbb{R}^n$. Then there* exists *a (possibly very small) positive number δ and a (scalar) function $y : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ that is a solution to the n^{th} order equation initial value problem*

$$\left\{ \begin{array}{ccc} y^{(n)} & = & f\left(x, y, y', ..., y^{(n-1)}\right) \\ y(x_0) & = & y_0^0 \\ y'(x_0) & = & y_0^1 \\ \vdots & \vdots & \vdots \\ y^{(n-1)}(x_0) & = & y_0^{n-1} \end{array} \right. . \tag{1.4}$$

If in addition there is $K > 0$ such that f satisfies the Lipschitz condition

$$|f(x, \mathbf{y}_1) - f(x, \mathbf{y}_2)| \leq K \|\mathbf{y}_1 - \mathbf{y}_2\|, \qquad \text{for all } (x_1, \mathbf{y}_1), (x_2, \mathbf{y}_2) \in \mathcal{R},$$

then this solution $y(x)$ is unique *in the sense that any two solutions must agree on their common interval of definition around x_0.*

2. Linear n^{th} order equations

We say that the n^{th} order equation (0.1) is *linear* if $f\left(x, y, y', ..., y^{(n-1)}\right)$ is an affine function of the variables $y, y', ..., y^{(n-1)}$ with coefficients that are functions of x, i.e.

$$\begin{aligned} & f\left(x, y, y', ..., y^{(n-1)}\right) \\ = \; & h(x) + f_0(x)\, y + +f_1(x)\, y' + ... + f_{n-1}(x)\, y^{(n-1)}, \end{aligned}$$

which we can rewrite in the form

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + ... + a_1(x)\,y' + a_0(x)\,y = g(x),$$

and more usually as

$$a_n(x)\frac{d^n}{dx^n}y + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}y + ... + a_1(x)\frac{d}{dx}y + a_0(x)\,y = g(x)\,. \tag{2.1}$$

We often abbreviate the left hand side by writing

$$L[y] = a_n(x)\frac{d^n}{dx^n}y + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}y + ... + a_1(x)\frac{d}{dx}y + a_0(x)\,y, \tag{2.2}$$

where it is understood that the coefficient functions $a_k(x)$ are associated with the linear operator L, which we refer to as a *linear n^{th} order differential operator*. We also refer to the function $g(x)$ on the right hand side of (2.1) as the *forcing* function.

The existence and uniqueness theory for linear equations is better behaved than in the general case because the Lipshitz condition is (locally) automatic, and in fact the solutions exist on the interval of definition of the coefficients $a_k(x)$, provided the top coefficient $a_n(x)$ doesn't vanish there.

THEOREM 7. *Suppose the functions $a_k(x)$ are continuous on an interval I for $0 \leq k \leq n$ and that $a_n(x) \neq 0$ for $x \in I$. Then the initial value problem*

$$\begin{cases} L[y] & = & g(x) \\ y(x_0) & = & y_0^0 \\ y'(x_0) & = & y_0^1 \\ \vdots & \vdots & \vdots \\ y^{(n-1)}(x_0) & = & y_0^{n-1} \end{cases} \tag{2.3}$$

has a unique solution $y : I \to \mathbb{R}$ defined and n times differentiable on the interval I.

The set of solutions to a linear equation has a great deal of structure that is missing in the general case, due mainly to the fact that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2] \tag{2.4}$$

for *any* two functions y_1 and y_2, and *any* two constants c_1 and c_2. This equation says that the operator L is a *linear* operator on functions, and its proof is immediate from the corresponding properties for derivatives, e.g.

$$(c_1y_1 + c_2y_2)' = c_1y_1' + c_2y_2'.$$

In fact, we have the following properties for solutions to the *homogeneous* equation $L[y] = 0$ (the word 'homogeneous' is used here just to indicate that the right hand side of the equation (2.1) vanishes).

CLAIM 1. *Let $L[y]$ be as in (2.2) with $a_k(x)$ continuous and $a_n(x)$ nonvanishing.*

(1) *If y_1 and y_2 are both solutions to the homogeneous equation $L[y] = 0$, and if c_1 and c_2 scalars, then the linear combination $y \equiv c_1y_1 + c_2y_2$ is also a solution. Indeed, if $L[y_1] = 0 = L[y_2]$, then (2.4) gives $L[c_1y_1 + c_2y_2] = 0$.*

(2) *Given an interval I, a point $x_0 \in I$, an operator L as in Theorem 7, and an integer $0 \leq k \leq n-1$, let $y_k(x)$ be the unique solution to the homogeneous initial value problem*

$$\left\{ \begin{array}{ccc} L[y] & = & 0 \\ y(x_0) & = & 0 \\ \vdots & \vdots & \vdots \\ y^{(k)}(x_0) & = & 1 \\ \vdots & \vdots & \vdots \\ y^{(n-1)}(x_0) & = & 0 \end{array} \right. ,$$

where $g(x) \equiv 0$, $y_k^{(j)}(x_0) = 0$ for $j \neq k$, and $y_k^{(k)}(x_0) = 1$. Then the unique solution $y(x)$ to the initial value problem (2.3) with $g(x) \equiv 0$ is given by

$$y = y_0^0 \; y_0(x) + y_0^1 \; y_1(x) + ... + y_0^{n-1} \; y_{n-1}(x), \qquad x \in I,$$

as can be seen by direct substitution. In other words, we simply take the linear combination of the special solutions $y_k(x)$ with constants equal to the specified initial conditions y_0^k!

(3) *The general solution of the homogeneous equation $L[y] = 0$ is given by the n-parameter family of solutions*

$$\{c_0 \; y_0(x) + c_1 \; y_1(x) + ... + c_{n-1} \; y_{n-1}(x)\}_{(c_0, ... c_{n-1}) \in \mathbb{R}^n} .$$

Indeed, each such function is seen to be a solution to $L[y] = 0$ upon applying (2.4) repeatedly. Conversely, if $y(x)$ satisfies $L[y] = 0$, then let $c_k = y^{(k)}(x_0)$ for $0 \leq k \leq n-1$. Then each of the functions $y(x)$ and $c_0 y_0(x) + ... + c_{n-1} y_{n-1}(x)$ satisfy the same *initial value problem, and so must be the same function by the uniqueness of solutions:*

$$\begin{aligned} & \frac{d^\ell}{dx^\ell} \{c_0 y_0 + ... + c_{n-1} y_{n-1}\}(x_0) \\ = \quad & c_0 y_0^{(\ell)}(x_0) + ... + c_{n-1} y_{n-1}^{(\ell)}(x_0) \\ = \quad & c_0 0 + ... + c_\ell y_\ell^{(\ell)}(x_0) + ... + c_{n-1} 0 \\ = \quad & c_\ell = y^{(\ell)}(x_0), \end{aligned}$$

for each $0 \leq \ell \leq n-1$.

Definition 2. *We say that a set of n functions $\mathcal{Y} = \{y_1, y_2, ..., y_n\}$ is a fundamental solution set for the homogeneous equation $L[y] = 0$ if the general solution is given by the n-parameter family*

$$\{c_1 y_1 + ... + c_n y_n\}_{(c_1, ... c_n) \in \mathbb{R}^n} .$$

Claim (3) above shows that there is *always* a fundamental solution set for the equation $L[y] = 0$, provided the coefficients of L are continuous and a_n is nonvanishing. Note that we are here relabeling the functions as $y_1, ..., y_n$ instead of $y_0, ..., y_{n-1}$.

Conclusion 1. *With a fundamental solution set $\mathcal{Y}$ to $L[y] = 0$ in hand, and together with just* one *particular solution y_p to the nonhomogeneous equation $L[y] =$*

g, the general solution *to the nonhomogeneous equation $L[y] = g$ is given by*

$$y = y_c + y_p; \qquad y_c = c_1 y_1 + ... + c_n y_n.$$

Here y_p is called a particular solution (to the nonhomogeneous equation) and y_c is called a complementary solution (to the homogeneous equation).

2.1. Second order linear equations. We divide the linear second order equation (2.1) (in which $n = 2$) through by the top coefficient $a_2(x)$, which we assume is nonvanishing. The result is this equation in which we have redefined the operator L and the forcing function $g(x)$:

$$L[y] \equiv y'' + p(x)\, y' + q(x)\, y = g(x)\,. \tag{2.5}$$

We cannot in general find solutions to this linear equation (although we can in the special case when the coefficient functions are constant), but if we are lucky or clever enough to find just *one* nontrivial solution to the associated homogeneous equation

$$L[y] = 0$$

(the trivial solution is $y(x) \equiv 0$), we can use the method of *reduction of order* to reduce the task of finding all the other solutions of (2.5), to that of solving a *first* order linear equation. The idea is to make two successive substitutions as follows.

Assume that $y_1(x)$ is a nontrivial solution to the *homogeneous* equation $L[y] = 0$. Make the substitution

$$y(x) = y_1(x)\, v(x)$$

in the *nonhomogeneous* equation $L[y] = g$, and compute the equation satisfied by the new unknown function $v(x)$. We expande $L[y_1 v]$, and regroup terms according to v and its derivatives, to get

$$\begin{aligned} g(x) &= L[y] = L[y_1 v] = (y_1 v)'' + p(x)(y_1 v)' + q(x)(y_1 v) \\ &= (y_1' v + y_1 v')' + p(x)(y_1' v + y_1 v') + q(x)(y_1 v) \\ &= (y_1'' v + 2y_1' v' + y_1 v'') + p(x)(y_1' v + y_1 v') + q(x)(y_1 v) \\ &= \{y_1'' + p(x)\, y_1' + q(x)\, y_1\}\, v + \{2y_1' + p(x)\, y_1\}\, v' + y_1 v'' \\ &= \{2y_1' + p(x)\, y_1\}\, v' + y_1 v'', \end{aligned} \tag{2.6}$$

since $y_1'' + p(x)\, y_1' + q(x)\, y_1 = L[y_1] = 0$. Now make the second substitution $z = v'$. Then z satisfies the linear equation

$$\begin{aligned} y_1(x)\, z' + \{2y_1'(x) + p(x)\, y_1(x)\}\, z &= g(x)\,; \\ z' + \left\{\frac{2y_1'}{y_1} + p\right\} z &= \frac{g}{y_1}. \end{aligned}$$

An integrating factor for this equation is

$$\mu = e^{\int \frac{2y_1'}{y_1} + p} = e^{2\ln|y_1| + \int p} = y_1^2 e^P \ ,$$

where $P = \int p$ is an antiderivative of p. Thus we have

$$\begin{aligned} (\mu z)' &= \left(y_1^2 e^P\right)' z + y_1^2 e^P z' = \mu \frac{g}{y_1} = y_1^2 e^P \frac{g}{y_1} = y_1 e^P g; \\ z &= y_1^{-2} e^{-P} \int y_1 e^P g. \end{aligned}$$

We can now antidifferentiate $z = v'$ to obtain v.

Finally, we obtain that $y \equiv y_1 v$ is a solution to the nonhomogeneous equation $L[y] = g$. Since there were two antiderivatives taken in the method above, there are two constants of integration in the formula for the solution y, and we have thus constructed a *two*-parameter family of solutions. This somewhat complicated procedure is best illustrated with an example.

EXAMPLE 14. *Given that $y_1(x) \equiv \frac{1}{x}$ solves the homogeneous equation*

$$2x^2y'' + 3xy' - y = 0,$$

we will use the method of reduction of order to find a general solution to the nonhomogeneous equation

$$2x^2y'' + 3xy' - y = x^3.$$

Let $y = y_1 v = \frac{1}{x}v$, and write the nonhomogeneous equation in standard form

$$y'' + \frac{3}{2x}y' - \frac{1}{2x^2}y = \frac{x}{2},$$

with $p(x) = \frac{3}{2x}$, $q(x) = -\frac{1}{2x^2}$ and $g(x) = \frac{x}{2}$. From (2.6) we have

$$\begin{aligned} \frac{x}{2} &= \{2y_1' + p(x)\, y_1\}\, v' + y_1 v'' \\ &= \left\{-\frac{2}{x^2} + \frac{3}{2x}\frac{1}{x}\right\} v' + \frac{1}{x}v'' \\ &= -\frac{1}{2x^2}v' + \frac{1}{x}v'', \end{aligned}$$

and so $z = v'$ satisfies

$$z' - \frac{1}{2x}z = \frac{x^2}{2}.$$

Then with $\mu = e^{-\int \frac{1}{2x}dx} = e^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}$, we obtain

$$\begin{aligned} \left(\frac{1}{\sqrt{x}}z\right)' &= (\mu z)' = \mu\frac{x^2}{2} = \frac{x^{\frac{3}{2}}}{2}; \\ \frac{1}{\sqrt{x}}z &= \int \frac{x^{\frac{3}{2}}}{2}dx = \frac{1}{5}x^{\frac{5}{2}} + c_2; \\ z &= \frac{1}{5}x^3 + c_2\sqrt{x}. \end{aligned}$$

Then we antidifferentiate $z = v'$ to get

$$v = \int z = \int \left(\frac{1}{5}x^3 + c_2\sqrt{x}\right) = \frac{1}{20}x^4 + \frac{2}{3}c_2 x^{\frac{3}{2}} + c_1,$$

and finally from $y = \frac{1}{x}v$ we obtain the two parameter family of solutions,

$$y = \frac{1}{20}x^3 + \frac{2}{3}c_2 x^{\frac{1}{2}} + c_1\frac{1}{x}, \qquad (c_1, c_2) \in \mathbb{R}^2.$$

It is worth noting that in the above example, the general solution has a very special form, namely

$$y = y_p + c_2 y_2 + c_1 y_1, \qquad (c_1, c_2) \in \mathbb{R}^2,$$

where $y_1 = \frac{1}{x}$ and $y_2 = \frac{2}{3}x^{\frac{1}{2}}$ are *different* solutions to the homogeneous equation

$$2x^2y'' + 3xy' - y = 0,$$

while $y_p = \frac{1}{20}x^3$ is a *particular* solution to the nonhomogeneous equation

$$2x^2y'' + 3xy' - y = x^3.$$

We now turn to making precise this notion that solutions are *different*, and this involves the concept of *linearly independent* functions.

2.2. Linear independence and Abel's formula. Two functions f and g defined on an interval I are said to be *linearly dependent on* I if one of them is a constant multiple of the other, i.e. either $f(x) = c_1 g(x)$, $x \in I$, for some scalar c_1 or $g(x) = c_2 f(x)$, $x \in I$, for some scalar c_2. The reason we consider both c_1 and c_2 is to include the possibility that one of the functions, say g, is identically zero, but not the other. Then we can write $g = 0 \cdot f$ but $f \neq c_1 g$ for any constant c_1. We can combine all possible cases in a single equation by simply requiring that there exist scalars c_1 and c_2 that are *not* both zero, such that

$$c_1 f(x) + c_2 g(x) = 0, \qquad x \in I.$$

The way to extend this concept to more than two functions is now evident. We say that a (finite) set of functions $\{f_1, f_2, ..., f_n\}$ is *linearly dependent on* I if there exist scalars $(c_1, c_2, ..., c_n) \neq (0, 0, ..., 0)$ such that

$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0, \qquad x \in I,$$

i.e. $c_1 f_1 + c_2 f_2 + ... + c_n f_n \equiv 0$ is the identically zero function on I.

Example 15. *The set of four functions*

$$\left\{1, \frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}, \frac{x^2}{(1-x)^2}\right\}$$

is linearly dependent on any interval I not containing 1, since with

$$(c_1, c_2, c_3, c_4) = (-1, 1, -2, 1),$$

we have

$$\begin{aligned} & c_1 1 + c_2 \frac{1}{(1-x)^2} + c_3 \frac{x}{(1-x)^2} + c_4 \frac{x^2}{(1-x)^2} \\ = \quad & -1 + \frac{1 - 2x + x^2}{(1-x)^2} = -1 + \frac{(1-x)^2}{(1-x)^2} = 0, \qquad x \neq 1. \end{aligned}$$

We say the set $\{f_1, f_2, ..., f_n\}$ is *linearly* ***in****dependent on* I if it is *not* linearly dependent on I. For a general collection of functions $\{f_1, f_2, ..., f_n\}$, the concept of linear independence cannot be easily characterized directly - it is simply the negation of linear dependence, which can be expressed as

$$c_1 f_1 + c_2 f_2 + ... + c_n f_n \equiv 0 \quad \Longrightarrow \quad c_1 = c_2 = ... = c_n = 0. \tag{2.7}$$

Example 16. *The set of three functions* $\{\sin x, \sin 2x, \sin 3x\}$ *is linearly independent on* $\mathbb{R}$ *. One of many ways to see that (2.7) holds is to assume that*

$$c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x \equiv 0,$$

and then let $x = \frac{\pi}{2}$ *to obtain*

$$c_1 - c_3 = c_1 \sin \frac{\pi}{2} + c_2 \sin \pi + c_3 \sin \frac{3\pi}{2} = 0.$$

Then differentiate the identity twice using $(\sin \omega x)'' = -\omega^2 \sin \omega x$ *to get*

$$-c_1 \sin x - 4c_2 \sin 2x - 9c_3 \sin 3x \equiv 0.$$

Then set $x = \frac{\pi}{2}$ *again to obtain*

$$-c_1 + 9c_3 = -c_1 \sin \frac{\pi}{2} - 4c_2 \sin \pi - 9c_3 \sin \frac{3\pi}{2} = 0.$$

Thus we have $9c_3 = c_1 = c_3$ *which gives* $c_3 = 0$*, then* $c_1 = 0$*, and then* $c_2 \sin 2x \equiv 0$ *gives* $c_2 = 0$*, and this completes the demonstration of the linear independence condition (2.7).*

But for special classes of functions we can do much better than verifying (2.7), especially when the functions $\{y_1, y_2, ..., y_n\}$ are *solutions* for the homogeneous equation $L[y] = 0$ on an interval I, where

$$L[y] = y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y \tag{2.8}$$

is a linear n^{th} order differential operator as defined in (2.2), but with $a_n \equiv 1$. Indeed, if $c_1 y_1 + c_2 y_2 + ... + c_n y_n \equiv 0$ on I, then we can differentiate both sides n times with respect to $x \in I$ to obtain

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) &= 0, \\ c_1 y_1'(x) + c_2 y_2'(x) + ... + c_n y_n'(x) &= 0, \\ &\vdots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + ... + c_n y_n^{(n-1)}(x) &= 0, \end{aligned}$$

for every $x \in I$. In matrix form this equation is

$$\begin{aligned} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} (x) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad x \in I; \\ \mathbf{M}(x)\mathbf{c} &= \mathbf{0}, \qquad x \in I; \\ \mathbf{Mc} &\equiv \mathbf{0}. \end{aligned}$$

We now conclude that the linear independence criterion (2.7) holds if and only if

$$\mathbf{Mc} \equiv \mathbf{0} \quad \Longrightarrow \quad \mathbf{c} = \mathbf{0}. \tag{2.9}$$

Note that so far we have only used that the functions $\{y_1, y_2, ..., y_n\}$ are $n-1$ times continuously differentiable, and we haven't yet exploited the fact that they are *solutions* to an n^{th} order linear homogeneous equation $L[y] = 0$.

To link (2.9) with the differential equation $L[y] = 0$, we recall from linear algebra that for a *fixed* $x \in I$,

$$\mathbf{M}(x)\mathbf{c} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{c} = \mathbf{0},$$

if and only if $\det \mathbf{M}(x) \neq 0$. We define this important determinant to be the *Wronskian* $\mathcal{W}(y_1, ..., y_n)(x)$ of the functions $\{y_1, y_2, ..., y_n\}$ on I:

$$\mathcal{W}(y_1, ..., y_n)(x) \equiv \det \mathbf{M}(x) = \det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}.$$

The connection of the Wronskian with a set of solutions is twofold. First we have a nonvanishing property from which we obtain the equivalence of linear independence and fundamental solution.

In the next three lemmas, the linear n^{th} order operator $L[y]$ is as in (2.8) with continuous coefficients $a_k(x)$ on an interval I.

LEMMA 1. *Suppose $\{y_1, y_2, ..., y_n\}$ is a set of solutions for the linear n^{th} order equation $L[y] = 0$ on I. Then the* Wronskian $\mathcal{W}(y_1, ..., y_n)(x)$ *is* ***either*** *identically zero on I,* ***or*** *never vanishing on I.*

LEMMA 2. *Suppose $\{y_1, y_2, ..., y_n\}$ is a set of solutions for the linear n^{th} order equation $L[y] = 0$ on I. Then the following four conditions are equivalent:*

(1) $\{y_1, y_2, ..., y_n\}$ is a fundamental solution set for $L[y] = 0$ on I,
(2) the *Wronskian* $\mathcal{W}(y_1, ..., y_n)(x)$ is nonzero for *all* $x \in I$,
(3) the *Wronskian* $\mathcal{W}(y_1, ..., y_n)(x_0)$ is nonzero for *some* $x_0 \in I$,
(4) the set $\{y_1, y_2, ..., y_n\}$ is linearly independent on I.

Second, if we work a bit harder, we can derive an explicit formula connecting the Wronskian $\mathcal{W}(y_1, ..., y_n)$ at two different points $x, x_0 \in I$, and that gives the conclusion of Lemma 1 above as an immediate corollary. This is Abel's formula involving the coefficient $a_{n-1}(x)$ of the linear differential operator L.

LEMMA 3 (Abel's formula). *Suppose $\{y_1, y_2, ..., y_n\}$ is a fundamental solution set for the linear n^{th} order equation $L[y] = 0$ on I. Then for $x, x_0 \in I$, we have*

$$\mathcal{W}(y_1, ..., y_n)(x) = e^{-\int_{x_0}^{x} a_{n-1}(s)ds}\mathcal{W}(y_1, ..., y_n)(x_0).$$

We will only give proofs in the simplest case $n = 2$, but will give them in a form which is not hard to generalize to the case $n > 2$. To establish Abel's formula, let y_1 and y_2 be two solutions of the second order linear homogenous equation

$$y'' + a_1(x)y' + a_0(x)y = 0.$$

We differentiate the Wronskian of y_1 and y_2 using the product rule for derivatives, and the multilinear and alternating properties of determinants:

$$\begin{aligned}
\frac{d}{dx}\mathcal{W}(y_1, y_2)(x) &= \frac{d}{dx}\det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \\
&= \det\begin{bmatrix} \frac{d}{dx}y_1(x) & \frac{d}{dx}y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} + \det\begin{bmatrix} y_1(x) & y_2(x) \\ \frac{d}{dx}y_1'(x) & \frac{d}{dx}y_2'(x) \end{bmatrix} \\
&= \det\begin{bmatrix} y_1'(x) & y_2'(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} + \det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1''(x) & y_2''(x) \end{bmatrix} \\
&= 0 + \det\begin{bmatrix} y_1(x) & y_2(x) \\ -a_1(x)y_1' - a_0(x)y_1(x) & -a_1(x)y_2' - a_0(x)y_2(x) \end{bmatrix} \\
&= \det\begin{bmatrix} y_1(x) & y_2(x) \\ -a_1(x)y_1' & -a_1(x)y_2' \end{bmatrix} + \det\begin{bmatrix} y_1(x) & y_2(x) \\ -a_0(x)y_1(x) & -a_0(x)y_2(x) \end{bmatrix} \\
&= -a_1(x)\mathcal{W}(y_1, y_2)(x) + 0.
\end{aligned}$$

This is both a linear and a separable equation for the Wronskian. Using the integrating factor $e^{\int_{x_0}^{x} a_1(s)ds}$ we get Abel's formula:

$$\begin{aligned} \frac{d}{dx}\left\{e^{\int_{x_0}^{x} a_1(s)ds}\mathcal{W}(y_1,y_2)(x)\right\} &= e^{\int_{x_0}^{x} a_1(s)ds}\left\{a_1(x)\mathcal{W}(y_1,y_2)(x)+\frac{d}{dx}\mathcal{W}(y_1,y_2)(x)\right\}=0; \\ \mathcal{W}(y_1,y_2)(x) &= Ce^{-\int_{x_0}^{x} a_1(s)ds}=\mathcal{W}(y_1,y_2)(x_0)\,e^{-\int_{x_0}^{x} a_1(s)ds}. \end{aligned}$$

Since the exponential function $e^{-\int_{x_0}^{x} a_1(s)ds}$ never vanishes, we immediately obtain Lemma 1 as a corollary.

We prove the equivalence of the four statements in Lemma 2 by linking the first to the second, and the third to the fourth. Of course the second and third are equivalent by Lemma 1.

We first show that (1) and (2) are equivalent. Indeed, (1) fails if and only if there is a triple (x_0,y_0,y_0') such that the solution $y(x)$ to the initial value problem

$$\left\{\begin{array}{ccc} y'' & = & -a_1y'-a_0y \\ y(x_0) & = & y_0 \\ y'(x_0) & = & y_0' \end{array}\right.$$

fails to have the form $y=c_1y_1+c_2y_2$ for any choice of scalars c_1 and c_2. Thus if and only if there is no solution $\mathbf{c}=(c_1,c_2)$ to the matrix equation

$$\mathbf{y}_0=\left[\begin{array}{c} y_0 \\ y_0' \end{array}\right]=\left[\begin{array}{cc} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]=\mathbf{M}(x_0)\,\mathbf{c},$$

i.e. the vector $\mathbf{y}_0=(y_0,y_0')$ is not in the range of the matrix $\mathbf{M}(x_0)$. It follows from linear algebra that this holds if and only if $\det\mathbf{M}(x_0)=0$. Since $\mathcal{W}(y_1,y_2)(x_0)=\det\mathbf{M}(x_0)$, this is equivalent to the failure of (2).

Now we demonstrate the equivalence of (3) and (4), beginning with (3) implies (4). Indeed, suppose that (3) holds, and in order to derive a contradiction, that (4) fails. Then by (3) there is some $x_0\in I$ such that $\det\mathbf{M}(x_0)\neq 0$, and by the failure of (4) there is a vector $\mathbf{c}\neq\mathbf{0}$ such that

$$\mathbf{0}=\left[\begin{array}{c} 0 \\ 0 \end{array}\right]=\left[\begin{array}{cc} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]=\mathbf{M}(x)\,\mathbf{c},$$

for all $x\in I$, and in particular for $x=x_0$. But $\det\mathbf{M}(x_0)\neq 0$ implies $\mathbf{c}=\mathbf{0}$ by linear algebra, and this is our desired contradiction. Conversely, we suppose that (3) fails and prove that (4) fails. Indeed, the failure of (3) shows that for any $x\in I$ we have $\mathcal{W}(y_1,y_2)(x)=0$, and in particular for a fixed $x_0\in I$. Then the equation

$$\mathbf{0}=\left[\begin{array}{c} 0 \\ 0 \end{array}\right]=\left[\begin{array}{cc} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]=\mathbf{M}(x_0)\,\mathbf{c},$$

has a solution $\mathbf{c}\neq\mathbf{0}$. Define the function $\varphi(x)\equiv c_1y_1(x)+c_2y_2(x)$. Then φ solves the initial value problem

$$\left\{\begin{array}{ccc} y'' & = & -a_1y'-a_0y \\ y(x_0) & = & 0 \\ y'(x_0) & = & 0 \end{array}\right.,$$

but so does the identically zero function $\mathbf{0}$! By the uniqueness of solutions, we conclude that $\varphi(x)=\mathbf{0}$, which says that the set $\{y_1,y_2\}$ is linearly dependent on I, hence that (4) fails.

Caution!: For functions $\{f_1, f_2\}$ that are not solutions to an equation, it remains true that if f_1 and f_2 are linearly dependent, then their Wronskian vanishes. But the converse is *false*: The pair of functions $\left\{x^3, |x|^3\right\}$ has vanishing Wronskian on the entire real line,

$$\mathcal{W}\left(x^3, |x|^3\right) = \det\begin{bmatrix} x^3 & |x|^3 \\ 3x^2 & 3x^2\frac{x}{|x|} \end{bmatrix} = 3x^5\frac{x}{|x|} - 3x^2|x|^3 \equiv 0,$$

but they are *not* linearly independent on any open interval containing the origin 0. **Exercise**: prove this!

2.2.1. *Using the Wronskian to find a second independent solution.* Now we show how to use Abel's formula to find a second linearly independent solution y_2 to the homogeneous equation $L[y] = 0$, if we are given a first nontrivial solution y_1. This provides a convenient alternative to carrying out the substitutions $z = v' = \left(\frac{y}{y_1}\right)'$ in the method of reduction of order. Of course, if we wish to solve the *non*homogenous equation $L[y] = g$, we must still use the substitutions $z = v' = \left(\frac{y}{y_1}\right)'$ in the method of reduction of order. We illustrate the use of Abel's formula by returning to the equation in Example 14.

Example 17. *Given that $y_1 = \frac{1}{x}$ solves the homogeneous equation*

$$L[y] \equiv y'' + \frac{3}{2x}y' - \frac{1}{2x^2}y = 0,$$

we fix any point x_0 other than the singular point 0, and assume that there is a second solution y_2 with $\mathcal{W}(y_1, y_2)(x_0) = 1$. Note that such a solution y_2 is automatically independent of y_1 by Lemma 2. Now we write out Abel's formula for the unknown function y_2:

$$\begin{aligned} \frac{1}{x}y_2' + \frac{1}{x^2}y_2 &= y_1y_2' - y_1'y_2 = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \\ &= \mathcal{W}(y_1, y_2)(x) = e^{-\int a_1(x)dx} = e^{-\int \frac{3}{2x}dx} \\ &= e^{-\frac{3}{2}\ln x} = x^{-\frac{3}{2}}. \end{aligned}$$

This equation is linear,

$$y_2' + \frac{1}{x}y_2 = x^{-\frac{1}{2}},$$

with integrating factor $\mu = x$, and so

$$\begin{aligned} (xy_2)' &= x\left(y_2' + \frac{1}{x}y_2\right) = xx^{-\frac{1}{2}} = x^{\frac{1}{2}}; \\ xy_2 &= \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}; \\ y_2 &= \frac{2}{3}x^{\frac{1}{2}}, \end{aligned}$$

gives a second independent solution y_2.

Note that we ignore the constant of integration here since we are only interested in finding the 'other' independent solution. Indeed, writing in the constant of integration C at the end would only give back a multiple of the known solution $\frac{1}{x}$, since $xy_2 = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C$ gives $y_2 = \frac{2}{3}x^{\frac{1}{2}} + C\frac{1}{x}$.

3. Constant coefficient linear equations

Now we consider the special case of the n^{th} order linear operator $L[y]$ in (2.2), where the coefficients $a_k(x)$ of the operator L are constants a_k:

$$L[y] \equiv a_n \frac{d^n}{dx^n} y + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y + ... + a_1 \frac{d}{dx} y + a_0 y. \tag{3.1}$$

In this case we will be able to give an explicit fundamental solution set to the homogeneous equation $L[y] = 0$, and develop very effective methods for solving the corresponding nonhomogeneous equations $L[y] = f$.

3.1. Fundamental solutions sets for the homogeneous case. The *first* order linear homogenous equation with constant coefficients $L[y] \equiv a_1 \frac{d}{dx} y + a_0 = 0$ is easily solved by separation of variables:

$$dy = -\frac{a_0}{a_1} dx; \qquad y = Ce^{-\frac{a_0}{a_1} x}.$$

In particular we note that the exponent $-\frac{a_0}{a_1}$ of the exponential is the unique *root* of the linear algebraic equation $a_1 r + a_0 = 0$. This suggests that we might search for a solution to the n^{th} order equation $L[y] = 0$ by plugging in exponential functions e^{rx} and see if they happen to satisfy the equation for certain exponents r. Since $\frac{d^k}{dx^k} e^{rx} = r^k e^{rx}$, we compute that

$$\begin{aligned} L[e^{rx}] &= a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + ... + a_1 r e^{rx} + a_0 e^{rx} \\ &= \left(a_n r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0\right) e^{rx}, \end{aligned}$$

which vanishes identically in x *if and only if* r is a root of the polynomial

$$P(r) = P_L(r) \equiv a_n r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0, \tag{3.2}$$

which we refer to as the *characteristic polynomial* of the constant coefficient linear n^{th} order operator L.

The fundamental theorem of algebra: It is a fundamental theorem in algebra that every polynomial P of order n has exactly n roots counting multiplicities in the field of complex numbers.

If r is a real root of P, then

$$y = Ce^{rx}$$

is a solution to the homogeneous equation $L[y] = 0$. If $r = a + ib$ is a complex root of P, then

$$y = Ce^{rx} = Ce^{(\alpha + ib)x} = Ce^{\alpha x} (\cos \beta x + i \sin \beta x)$$

by deMoivres formula $e^{i\theta} = \cos\theta + i \sin\theta$ and properties of exponents. Thus we have produced a complex-valued solution $y(x) = e^{rx}$ to the differential equation $L[y] = 0$. We can write this solution uniquely in terms of its real and imaginary parts as

$$y(x) \equiv u(x) + iv(x),$$

where both $u(x) = \operatorname{Re} y(x)$ and $v(x) = \operatorname{Im} y(x)$ are real-valued functions. If the coefficients a_k are *real* numbers, then it is easy to see that both $u(x)$ and $v(x)$ are themselves solutions to the homogeneous equation:

$$\begin{aligned}
L[u](x) &= a_n \frac{d^n}{dx^n} u(x) + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} u(x) + \dots + a_1 \frac{d}{dx} u + a_0 u(x) \\
&= a_n \frac{d^n}{dx^n} \operatorname{Re} y(x) + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} \operatorname{Re} y(x) + \dots + a_1 \frac{d}{dx} \operatorname{Re} y(x) + a_0 \operatorname{Re} y(x) \\
&= \operatorname{Re} \left(a_n \frac{d^n}{dx^n} y(x) + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y(x) + \dots + a_1 \frac{d}{dx} y + a_0 y(x) \right) \\
&= \operatorname{Re} L[y](x) = \operatorname{Re} 0 = 0,
\end{aligned}$$

and similarly $L[v](x) = 0$. We can also easily compute that

$$\begin{aligned}
u(x) &= \operatorname{Re} y(x) = \operatorname{Re} e^{rx} = \operatorname{Re} e^{(\alpha+i\beta)x} = \operatorname{Re} \{ e^{\alpha x} (\cos \beta x + i \sin \beta x) \} = e^{\alpha x} \cos \beta x; \\
v(x) &= \operatorname{Im} y(x) = \operatorname{Im} e^{rx} = \operatorname{Im} e^{(\alpha+i\beta)x} = \operatorname{Im} \{ e^{\alpha x} (\cos \beta x + i \sin \beta x) \} = e^{\alpha x} \sin \beta x.
\end{aligned}$$

Thus in this case we have produced *two* linearly independent solutions,

$$\begin{aligned}
y_1(x) &= e^{\alpha x} \cos \beta x, \\
y_2(x) &= e^{\alpha x} \sin \beta x,
\end{aligned}$$

to the homogeneous equation $L[y] = 0$. They are linearly independent because $\beta \neq 0$ by assumption that the root $r = \alpha + i\beta$ is complex, and their Wronskian is *nonzero*:

$$\begin{aligned}
\mathcal{W}(y_1, y_2)(x) &= \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \\
&= \det \begin{bmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ a e^{\alpha x} \cos \beta x - \beta e^{\alpha x} \sin \beta x & a e^{\alpha x} \sin \beta x + \beta e^{\alpha x} \cos \beta x \end{bmatrix} \\
&= e^{2\alpha x} \det \begin{bmatrix} \cos \beta x & \sin \beta x \\ -\beta \sin \beta x & \beta \cos \beta x \end{bmatrix} \\
&= e^{2\alpha x} \beta \left(\cos^2 \beta x + \sin^2 \beta x \right) = \beta e^{2\alpha x}.
\end{aligned}$$

REMARK 2. *Note that in the case the coefficients a_k of L are all real numbers, then the complex roots appear in complex conjugate pairs: $\alpha + i\beta$ is a root of the characteristic polynomial $P(r)$ in (3.2) if and only if $\alpha - i\beta$ is root since*

$$\begin{aligned}
\overline{P(\alpha + i\beta)} &= a_n \overline{(\alpha + i\beta)^n} + a_{n-1} \overline{(\alpha + i\beta)^{n-1}} + \dots + a_1 \overline{(\alpha + i\beta)} + a_0 \\
&= a_n (\alpha - i\beta)^n + a_{n-1} (\alpha - i\beta)^{n-1} + \dots + a_1 (\alpha - i\beta) + a_0 = P(\alpha - i\beta).
\end{aligned}$$

Thus it is not surprising that a complex root delivers two independent solutions - namely 'one for each root in the complex pair'.

EXAMPLE 18. *The general solution of the equation*

$$y'' + 5y' + 6y = 0$$

is found by computing the roots of the associated characteristic polynomial

$$0 = r^2 + 5r + 6 = (r + 2)(r + 3).$$

The roots are $r = -2, -3$, and the general solution is then given by

$$y = c_1 e^{-2x} + c_2 e^{-3x}.$$

EXAMPLE 19. *The solution to the initial value problem*

$$\begin{cases} y'' - 2y' + 5y &= 0 \\ y(0) &= 1 \\ y'(0) &= 2 \end{cases},$$

can be found as follows. First, find the general solution of the differential equation:

$$\begin{aligned} 0 &= r^2 - 2r + 5; \quad r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i; \\ y &= c_1 e^x \cos 2x + c_2 e^x \sin 2x. \end{aligned}$$

Second, differentiate y to get

$$y' = c_1 e^x (\cos 2x - 2\sin 2x) + c_2 e^x (\sin 2x + 2\cos 2x),$$

and then substitute these formulas into the initial conditions to solve for the constants c_1, c_2:

$$\begin{aligned} 1 &= y(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1; \\ 2 &= c_1 (1 - 2 \cdot 0) + c_2 (0 + 2 \cdot 1) = c_1 + 2c_2. \end{aligned}$$

This gives $c_1 = 1$, $c_2 = \frac{1}{2}$, and the solution to the initial value problem is hence

$$y = e^x \cos 2x + \frac{1}{2} e^x \sin 2x.$$

EXAMPLE 20. *Try to find the general solution to the equation*

$$y'' - 6y' + 9y = 0.$$

The characteristic polynomial is $r^2 - 6r + 9 = (r-3)^2$, and has a single repeated root 3. Thus we know that

$$y_1 = c_1 e^{3x}$$

is a solution but this method doesn't produce a second independent solution. Of course we could take $y_1 = e^{3x}$ in Abel's formula, and solve for y_2:

$$\begin{aligned} y_1 y_2' - y_1' y_2 &= \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \mathcal{W}(y_1, y_2)(x) = e^{6x} \mathcal{W}(y_1, y_2)(x_0) = e^{6x}; \\ e^{3x} y_2' - 3e^{3x} y_2 &= e^{6x}; \\ \left(e^{-3x} y_2\right)' &= e^{-3x} (y_2' - 3y_2) = e^{-3x} e^{3x} = 1; \\ e^{-3x} y_2 &= x + C; \\ y_2 &= x e^{3x} + C e^{3x}. \end{aligned}$$

Thus we have found a second independent solution $y_2 = xe^{3x}$, and the general solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} = (c_1 + c_2 x) e^{3x}.$$

The three examples above suggest the following theorem.

THEOREM 8. *Let $L[y]$ be a constant coefficient n^{th} order linear differential operator as in (3.1). Suppose that the characteristic polynomial $P(r)$ of the (real) constant coefficient n^{th} order linear differential operator $L[y]$ factors as*

$$\begin{aligned} &P(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \ldots (r - r_M)^{k_M} \\ &\times \{(r - [\alpha_1 + i\beta_1])(r - [\alpha_1 - i\beta_1])\}^{\ell_1} \ldots \{(r - [\alpha_N + i\beta_N])(r - [\alpha_N - ib\beta_N])\}^{\ell_N}, \end{aligned}$$

where the r_j, α_j, β_j are all real numbers, and

$$n = k_1 + ... + k_M + 2\ell_1 + ... + 2\ell_N.$$

Thus each r_j is a real root of multiplicity k_j and each pair of complex conjugate roots $(\alpha_j + i\beta_j, \alpha_j - i\beta_j)$ has multiplicity ℓ_j. Then the set of n functions

$$\begin{aligned}
&e^{r_1 x},\ xe^{r_1 x}, ...x^{k_1-1}e^{r_1 x};\\
&e^{r_2 x},\ xe^{r_2 x}, ...x^{k_2-1}e^{r_2 x};\\
&\vdots\\
&e^{r_M x},\ xe^{r_M x}, ...x^{k_M-1}e^{r_M x};\\
&e^{\alpha_1 x}\cos\beta_1 x,\ xe^{\alpha_1 x}\cos\beta_1 x, ...x^{k_1-1}e^{\alpha_1 x}\cos\beta_1 x;\\
&e^{\alpha_1 x}\sin\beta_1 x,\ xe^{\alpha_1 x}\sin\beta_1 x, ...x^{k_1-1}e^{\alpha_1 x}\sin\beta_1 x;\\
&\vdots\\
&e^{\alpha_N x}\cos\beta_N x,\ xe^{\alpha_N x}\cos\beta_N x, ...x^{\ell_N-1}e^{\alpha_N x}\cos\beta_{N1} x;\\
&e^{\alpha_N x}\sin\beta_N x,\ xe^{\alpha_N x}\sin\beta_N x, ...x^{\ell_N-1}e^{\alpha_N x}\sin\beta_N x,
\end{aligned} \tag{3.3}$$

is a fundamental solution set for the homogeneous equation $L[y] = 0$, and the general solution is given by

$$\begin{aligned}
y &= \sum_{j=1}^{M}\left\{\sum_{s=1}^{k_j} c_{j,s}\ x^{s-1}\right\}\ e^{r_j x}\\
&\quad + \sum_{j=1}^{N}\left\{\sum_{s=1}^{\ell_j}\left(d_{j,s}\ \cos\beta_j x + e_{j,s}\ \sin\beta_j x\right)\ x^{s-1}\right\}\ e^{\alpha_j x}\ ,
\end{aligned}$$

where $c_{j,s}$, $d_{j,s}$ and $e_{j,s}$ are real constants.

Here is how we can informally remember the conclusion of this theorem:

(1) For each repetition of a real root r_j, there is a solution

$$x^{s-1}e^{r_j x},$$

with $1 \le s \le k_j$, and

(2) for each repetition of a complex conjugate pair of roots $\alpha_j \pm i\beta_j$, there are two solutions

$$x^{s-1}e^{\alpha_j x}\cos\beta_j x \text{ and } x^{s-1}e^{\alpha_j x}\sin\beta_j x,$$

with $1 \le s \le \ell_j$.

PROBLEM 5. *Find a fundamental solution set for the equation*

$$y^{(4)} - y''' - 3y'' + 5y' - 2y = 0.$$

SOLUTION 5. *The characteristic polynomial is,*

$$P(r) = r^4 - r^3 - 3r^2 + 5r - 2.$$

To factor this polynomial, we hope there is a rational root $\frac{p}{q}$ in reduced form, in which case we must have that p divides the constant coefficient -2, and that q

divides the highest power coefficient 1*. Thus our only choices for a rational root are* $\pm 1, \pm 2$*. Substitution shows that*

$$P(1) = 0, \quad P(-1) = -8, \quad P(2) = 4, \quad P(-2) = 0,$$

and so we know that

$$(r-1)(r+2) = r^2 + r - 2$$

is a factor of $P(r)$*. We now apply the long division algorithm due to Euclid to get*

$$\begin{array}{lcllll} & & r^2 & -2r & +1 & \\ r^2+r-2 & \rightarrow & r^4 - r^3 - 3r^2 + 5r - 2 & & & \\ & & r^4 + r^3 - 2r^2 & & & \\ & & & -2r^3 - r^2 + 5r - 2 & & . \\ & & & -2r^3 - 2r^2 + 4r & & \\ & & & & r^2 + r - 2 & \end{array}$$

Thus we have the factorization

$$\begin{aligned} P(r) &= (r-1)(r+2)\left(r^2 - 2r + 1\right) \\ &= (r-1)^3 (r+2), \end{aligned}$$

and guided by point (1) above, Theorem 8 gives the fundamental solution set

$$\left\{e^x,\ xe^x,\ x^2 e^x,\ e^{-2x}\right\}.$$

PROBLEM 6. *Find a fundamental solution set for the equation*

$$y^{(4)} - 8y''' + 26y'' - 40y' + 25y = 0,$$

given that its characteristic polynomial factors as

$$r^4 - 8r^3 + 26r^2 - 40r + 25 = \left(r^2 - 4r + 5\right)^2.$$

SOLUTION 6. *The roots of the quadratic polynomial* $r^2 - 4r + 5$ *are*

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = 2 \pm i.$$

Guided by point (2) above, Theorem 8 gives the fundamental solution set

$$\left\{e^{2x}\cos x,\ e^{2x}\sin x,\ xe^{2x}\cos x,\ xe^{2x}\sin x\right\}.$$

3.2. Undetermined coefficients in the nonhomogeneous case. Now we consider the nonhomogeneous equation

$$L[y] = f,$$

where L is a constant coefficient n^{th} order linear differential operator as in (3.1),

$$L[y] \equiv a_n \frac{d^n}{dx^n} y + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y + \ldots + a_1 \frac{d}{dx} y + a_0.$$

We assume the forcing function f is a finite linear combination of the type of functions arising in the fundamental solutions sets to *arbitrary* homogeneous linear constant coefficient equations, namely functions of the form:

$$x^k e^{\alpha x} \cos \beta x,\ x^k e^{\alpha x} \sin \beta x, \qquad k = 0, 1, 2, \ldots \text{ and } \alpha, \beta \in \mathbb{R}.$$

Such functions include all polynomials $Q(x)$ in x,

$$Q(x) = b_M x^M + b_{M-1} x^{M-1} + b_1 x + b_0,$$

and all products $Q(x)e^{\alpha x}$ of polynomials $Q(x)$ with exponentials $e^{\alpha x}$ and sines and cosines $\cos\beta x$ and $\sin\beta x$, and of course sums of such. We will denote the set of linear combinations of these functions by $\mathcal{F}$:

$$\mathcal{F} \equiv \left\{ \sum_{j=1}^{M} Q_j(x)\, e^{\alpha_j x} \cos\beta_j x + R_j(x)\, e^{\alpha_j x} \sin\beta_j x \right\},$$

where the sums are taken over all $M = 0, 1, 2, ...$; and where $\alpha_j, \beta_j \in \mathbb{R}$, and Q_j and R_j are polynomials in x. Remember that we can take the polynomials to be the constant 1, and the numbers α_j, β_j to be 0, so that functions like $e^{5x}\cos 2x + x^3 \sin 4x$ belong to $\mathcal{F}$.

There are two important properties of this vector space of functions

(1) $\mathcal{F}$ contains the general solutions of all homogeneous n^{th} order linear differential equations with constant coefficients;
(2) $\mathcal{F}$ is closed under the operation of differentiation, namely, if $f \in \mathcal{F}$ then $f' \in \mathcal{F}$, which we can abbreviate by writing

$$\frac{d}{dx} : \mathcal{F} \to \mathcal{F}.$$

To see property (2), we simply note that the product rule, together with formulas for derivatives of elementary functions, give

$$\begin{aligned} &\frac{d}{dx}\left(x^k e^{\alpha x} \cos\beta x\right) \\ &\qquad = k x^{k-1} e^{\alpha x} \cos\beta x + x^k \alpha e^{x} \cos\beta x - x^k e^{\alpha x} \beta \sin\beta x \in \mathcal{F}; \\ &\text{and similarly } \frac{d}{dx}\left(x^k e^{\alpha x} \sin\beta x\right) \in \mathcal{F}. \end{aligned}$$

Of course it now follows that

$$\frac{d^k}{dx^k} : \mathcal{F} \to \mathcal{F}, \qquad k = 0, 1, 2, ...$$

Conclusion 2. *From the facts that $\mathcal{F}$ contains all solutions y_c to the homogeneous equation*

$$L[y] = 0,$$

and is closed under repeated differentiation, it is reasonable to conclude that if the forcing function f lies in $\mathcal{F}$, then a particular solution y_p to the nonhomogeneous equation

$$L[y] = f,$$

also lies in $\mathcal{F}$. It remains only to narrow our search by making an intelligent guess with undetermined coefficients *for the form of y_p, and then to plug our guess for y_p into the equation $L[y] = f$ to determine the coefficients.*

In order to motivate our scheme for intelligently guessing the form of a particular solution y_p, we look at a few examples first.

Example 21. *In order to solve the equation*

$$y'' + 4y = 5x^2 e^x,$$

we assume there is a particular solution $y_p \in \mathcal{F}$*, and since* $f = 5x^2e^x$ *and all of its derivatives are linear combinations of* x^2e^x*,* xe^x *and* e^x*, we might guess that* y_p *has the special form*

$$y_p(x) = Ax^2e^x + Bxe^x + Ce^x = \left\{Ax^2 + Bx + C\right\} e^x$$

where A*,* B *and* C *are* undetermined coefficients*. Then*

$$\begin{aligned} y_p'(x) &= \{2Ax + B\} e^x + \left\{Ax^2 + Bx + C\right\} e^x \\ &= \left\{Ax^2 + (2A + B)x + (B + C)\right\} e^x; \\ y_p''(x) &= \{2Ax + (2A + B)\} e^x + \left\{Ax^2 + (2A + B)x + (B + C)\right\} e^x \\ &= \left\{Ax^2 + (4A + B)x + (2A + 2B + C)\right\} e^x. \end{aligned}$$

We now plug this form into the equation and use the above expressions for y_p'' *and* y_p *to get*

$$\begin{aligned} 5x^2e^x &= y_p'' + 4y_p \\ &= \left\{Ax^2 + (4A + B)x + (2A + 2B + C)\right\} e^x + 4\left\{Ax^2 + Bx + C\right\} e^x \\ &= (5A)x^2e^x + (4A + 5B)xe^x + (2A + 2B + 5C)e^x. \end{aligned}$$

Since the functions $\left\{x^2e^x, xe^x, e^x\right\}$ *are linearly independent, the above identity gives equality of the coefficients:*

$$\begin{aligned} 5A &= 5, \\ 4A + 5B &= 0, \\ 2A + 2B + 5C &= 0, \end{aligned}$$

from which we obtain

$$A = 1, B = -\frac{4}{5}, C = -\frac{2}{25}.$$

Thus a particular solution is

$$y_p(x) = \left(x^2 - \frac{4}{5}x - \frac{2}{25}\right) e^x.$$

Example 22. *To solve the equation*

$$y'' + 3y' + 2y = \sin x,$$

we note that $f = \sin x$ *and all of its derivatives are linear combinations of* $\sin x$ *and* $\cos x$*, and so we might guess that* y_p *has the special form*

$$y_p(x) = A\sin x + B\cos x,$$

where A *and* B *are* undetermined coefficients*. Then*

$$\begin{aligned} y_p'(x) &= A\cos x - B\sin x; \\ y_p''(x) &= -A\sin x - B\cos x. \end{aligned}$$

We now plug this form into the equation and use the above expressions for y_p''*,* y_p' *and* y_p *to get*

$$\begin{aligned} \sin x &= y_p'' + 3y_p' + 2y_p \\ &= (-A\sin x - B\cos x) \\ &\quad +3(A\cos x - B\sin x) + 2(A\sin x + B\cos x) \\ &= (A - 3B)\sin x + (B + 3A)\cos x. \end{aligned}$$

Since $\{\sin x, \cos x\}$ *are linearly independent, we can equate coefficients to get*

$$\begin{aligned} A - 3B &= 1, \\ B + 3A &= 0, \end{aligned}$$

from which we obtain

$$A = \frac{1}{10}, B = -\frac{3}{10}.$$

Thus a particular solution is

$$y_p(x) = \frac{1}{10}\sin x - \frac{3}{10}\cos x = \frac{\sin x - 3\cos x}{10}.$$

Thus it appears that a general rule is emerging from these two examples, namely that given $f \in \mathcal{F}$, a particular solution y_p has the same form as that of f and all of its derivatives. This is not quite true as the next example shows.

Example 23. *To find a particular solution of the equation*

$$y'' + 2y' + y = 5xe^{-x},$$

we might guess, based on our experience with the first two examples, that a particular solution has the form

$$y_p(x) = Axe^{-x} + Be^{-x},$$

since $f = 5xe^{-x}$ *and all of its derivatives are linear combinations of* xe^{-x} *and* Be^{-x}. *But clearly this is not the case! Both of the functions* xe^{-x} *and* e^{-x} *are solutions to the homogeneous equation* $L[y] = 0$, *hence*

$$L[y_p] = A + BL\left[e^{-x}\right] = 0 + 0 = 0.$$

This is the first time we have come across the situation where the forcing function is (or includes) a solution of the homogeneous equation. The 'trick' is to assume that y_p *has the special form*

$$y_p = x^2\left(Axe^{-x} + Be^{-x}\right),$$

where the exponent of the extra power of x *is chosen to be the multiplicity of the corresponding root in the characteristic polynomial,*

$$P(r) = r^2 + 2r + 1 = (r+1)^2.$$

Then

$$\begin{aligned} y_p &= \left\{Ax^3 + Bx^2\right\}e^{-x}; \\ y_p' &= \left\{3Ax^2 + 2Bx\right\}e^{-x} - \left\{Ax^3 + Bx^2\right\}e^{-x} \\ &= \left\{-Ax^3 + (3A - B)x^2 + 2Bx\right\}e^{-x}; \\ y_p'' &= \left\{-3Ax^2 + (6A - 2B)x + 2B\right\}e^{-x} \\ &\quad - \left\{-Ax^3 + (3A - B)x^2 + 2Bx\right\}e^{-x} \\ &= \left\{Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B\right\}e^{-x}. \end{aligned}$$

We now plug this form into the equation and use the above expressions for y_p'', y_p' *and* y_p *to get*

$$\begin{aligned} 5xe^{-x} &= y_p'' + 2y_p' + y_p \\ &= \{6Ax + 2B\}e^{-x}. \end{aligned}$$

Since $\{xe^{-x}, e^{-x}\}$ *are linearly independent, we can equate coefficients to get*

$$6A = 5, 2B = 0,$$

from which we obtain

$$A = \frac{5}{6}, B = 0.$$

Thus a particular solution is

$$y_p(x) = \frac{5}{6}x^3e^{-x}.$$

One final comment. The linearity of the operator L shows that if $f = g + h$ is a sum of forcing functions g and h, and if y_p and z_p solve the nonhomogeneous equations $L[y_p] = g$ and $L[z_p] = h$ respectively, then the sum $y_p + z_p$ of the particular solutions is a particular solution to the nonhomogeneous equation

$$L[y_p + z_p] = L[y_p] + L[z_p] = g + h.$$

For example, we showed above that with $L[y] = y'' + 4y$, the function $y_p(x) = \left(x^2 - \frac{4}{5}x - \frac{2}{25}\right)e^x$ satisfies $L[y_p] = 5x^2e^x$. We also note that a simple calculation shows that $z_p \equiv e^{-x}$ satisfies $L[z_p] = 5e^{-x}$. Hence by linearity of L, we have

$$\begin{aligned} L\left[\left(x^2 - \frac{4}{5}x - \frac{2}{25}\right)e^x + e^{-x}\right] &= L[y_p + z_p] \\ &= L[y_p] + L[z_p] = 5x^2e^x + 5e^{-x}, \end{aligned}$$

and so $\left(x^2 - \frac{4}{5}x - \frac{2}{25}\right)e^x + e^{-x}$ is a particular solution of the nonhomogeneous equation $L[y] = 5\left(x^2e^x + e^{-x}\right)$.

The examples above suggest the following theorem.

THEOREM 9. *Let* $L[y]$ *be a constant coefficient* n^{th} *order linear differential operator as in (3.1), and consider the nonhomogeneous equation*

$$L[y] = x^ke^{\alpha x}\cos\beta x \text{ or } x^ke^{\alpha x}\sin\beta x,$$

where $k = 0, 1, 2, ...$ *and* $\alpha, \beta \in \mathbb{R}$. *A particular solution* y_p *is given by*

$$y_p = x^s\{Q(x)e^{\alpha x}\cos\beta x + R(x)e^{\alpha x}\sin\beta x\},$$

where $Q(x) = \sum_{j=0}^{k} B_jx^j$ *and* $R(x) = \sum_{j=0}^{k} C_jx^j$ *are polynomials in* x *of degree* k, *and with* undetermined coefficients B_j *and* C_j *which can be determined by substitution in the equation, and finally, where* s *is the multiplicity of the root* $\alpha + i\beta$ *in the characteristic polynomial* $P(r)$ *associated with* $L[y]$ *(if* $\alpha + i\beta$ *is not a root, then its multiplicity is* 0*).*

EXAMPLE 24. *The form of a particular solution* y_p *to the equation*

$$y'' + 2y' + 2y = e^{-x}\cos x + x^2$$

is

$$y_p = x^1\left(Be^{-x}\cos x + Ce^{-x}\sin x\right) + A_2x^2 + A_1x + A_0,$$

since the characteristic polynomial r^2+2r+2 *has a conjugate pair of roots* $\frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$ *with multiplicity* $s = 1$.

EXAMPLE 25. *The form of a particular solution y_p to the equation*

$$y''' - 2y'' + y' = x - e^x$$

is

$$y_p = x^1 (A_0 + A_1 x) + x^2 (Be^x) = A_0 x + A_1 x^2 + Bx^2 e^x,$$

since the characteristic polynomial

$$r^3 - 2r^2 + r = r (r - 1)^2$$

has root $r = 0$ with multiplicity $s = 1$, and root $r = 1$ with multiplicity $s = 2$.

4. Variation of parameters

Suppose L is an n^{th} order linear differential operator as in (2.2). We present a general method, called *variation of parameters*, that applies even in the *variable* coefficient case, for solving the nonhomogeneous equation

$$L[y] = g,$$

provided we know a fundamental solution set for the associated homogeneous equation $L[y] = 0$. In the special case that L has constant coefficients, then we do indeed know a fundamental solution set for $L[y] = 0$ (if we can factor the characteristic polynomial), and so the method of variation of parameters always applies in this case, and for *any* choice of forcing function, not just those in the space $\mathcal{F}$ introduced in the previous section.

The idea in this method is to consider a fundamental solution set $\{y_1, ..., y_n\}$ to $L[y] = 0$, and replace the constants c_j in the general solution to $L[y] = 0$,

$$y_c (x) = c_1 y_1 (x) + ... + c_n y_n (x),$$

with *functions* $v_j (x)$ in the hope that for some choice of v_j*'s* the function

$$y_p (x) = v_1 (x) y_1 (x) + ... + v_n (x) y_n (x)$$

will be a particular solution to the nonhomogeneous equation $L[y] = g$. This does indeed work, and the functions v_j can be determined by substitution in the nonhomogeneous equation, followed by quite lengthy calculations. In order to minimize both the effort and the chance of computational error, we can greatly streamline the organization of the calculations by returning to the connection between n^{th} order equations and first order $n \times n$ systems as discussed earlier. This approach also proves that the method will always succeed.

Recall that $y(x)$ solves

$$y^{(n)} + a_{n-1} (x) y^{(n-1)} + ... + a_1 (x) y' + a_0 (x) y \equiv L[y] = g(x),$$

if and only if the vector function

$$\mathbf{y} \equiv \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{bmatrix}$$

solves the $n \times n$ system

$$\begin{aligned}\mathbf{y}'(x) &= \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ -a_0(x) & -a_1(x) & & & -a_{n-1}(x) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(x) \end{bmatrix} \\ &\equiv M(x)\,\mathbf{y}(x) + \mathbf{g}(x),\end{aligned}$$

where $\mathbf{g}(x) \equiv \begin{bmatrix} 0 \\ \vdots \\ g(x) \end{bmatrix}$. If $\{\varphi_1, ..., \varphi_n\}$ is a fundamental solution set for $L[y] = 0$, then each vector function $\boldsymbol{\varphi}_j \equiv \begin{bmatrix} \varphi_j(x) \\ \vdots \\ \varphi_j^{(n-1)}(x) \end{bmatrix}$ solves the equation $\boldsymbol{\varphi}_j'(x) = M(x)\,\boldsymbol{\varphi}_j(x)$, and so if we arrange the columns $\boldsymbol{\varphi}_j$ into a matrix

$$\boldsymbol{\Phi} \equiv \begin{bmatrix} \varphi_1(x) & \cdots & \varphi_n(x) \\ \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(x) & \cdots & \varphi_n^{(n-1)}(x) \end{bmatrix},$$

we get the matrix equation

$$\boldsymbol{\Phi}'(x) = M(x)\,\boldsymbol{\Phi}(x).$$

In the simple case $n = 2$ we can write this out in full as

$$\frac{d}{dx}\begin{bmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0(x) & -a_1(x) \end{bmatrix}\begin{bmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{bmatrix}.$$

Now we treat matrices like numbers, and with $\mathbf{v} \equiv \begin{bmatrix} v_1(x) \\ \vdots \\ v_n(x) \end{bmatrix}$, we compute using the product rule that

$$\begin{aligned}(\boldsymbol{\Phi}(x)\,\mathbf{v}(x))' &= \boldsymbol{\Phi}'(x)\,\mathbf{v}(x) + \boldsymbol{\Phi}(x)\,\mathbf{v}'(x) \\ &= M(x)\,\boldsymbol{\Phi}(x)\,\mathbf{v}(x) + \boldsymbol{\Phi}(x)\,\mathbf{v}'(x).\end{aligned}$$

Thus the vector function $\mathbf{y}(x) = \boldsymbol{\Phi}(x)\,\mathbf{v}(x)$ satisfies the system

$$\mathbf{y}'(x) = M(x)\,\mathbf{y}(x) + \mathbf{g}(x),$$

if and only if the derivative $\mathbf{v}'(x)$ of the vector function $\mathbf{v}(x)$ satisfies

$$\boldsymbol{\Phi}(x)\,\mathbf{v}'(x) = \mathbf{g}(x).$$

If we can solve this latter equation, then the first component $y(x)$ of the vector $\mathbf{y}(x) = \boldsymbol{\Phi}(x)\,\mathbf{v}(x)$, which is

$$y(x) = \varphi_1(x)\,v_1(x) + ... + \varphi_n(x)\,v_n(x),$$

is a solution to the nonhomogeneous n^{th} order equation

$$L[y] = g.$$

But the determinant of the matrix $\mathbf{\Phi}(x)$ is precisely the Wronskian $\mathcal{W}(\varphi_1,...,\varphi_n)(x)$ of the solutions in the fundamental solution set, so is *nonvanishing* for all x! Thus the matrix $\mathbf{\Phi}(x)$ is invertible for all x, and we can solve for $\mathbf{v}'(x)$ and then $\mathbf{v}(x)$:

$$\begin{aligned}\mathbf{v}'(x) &= \mathbf{\Phi}(x)^{-1}\mathbf{g}(x);\\ \mathbf{v}(x) &= \int \mathbf{\Phi}(x)^{-1}\mathbf{g}(x)\,dx.\end{aligned}$$

Remark 3. *This latter formula*

$$\mathbf{v}(x) = \int \mathbf{\Phi}(x)^{-1}\mathbf{g}(x)\,dx,$$

is particularly easy to remember, along with the fact that the first component in $\mathbf{\Phi}(x)\mathbf{v}(x)$ *is then a particular solution* y_p *to the nonhomogeneous equation* $L[y] = g$. *In fact, in the order one case* $y' + a_0 y = g$, *we see that this formula is the usual integrating factor formula,*

$$\mu(x)\,y(x) = \int \mu(x)\,g(x)\,dx,$$

once we have observed that the integrating factor $\mu(x) = e^{\int a_0(x)dx}$ *is the* reciprocal *of the solution* $\varphi(x) = e^{-\int a_0(x)dx}$ *to the homogeneous equation* $y' + a_0 y = 0$; *then* $v = \mu y$ *and* $\Phi^{-1} = \frac{1}{\varphi} = \mu$.

4.1. The second order case. In the special case $n = 2$ these calculations can be written out in full, making them perhaps more transparent. Repeating the above with $n = 2$ we have from the product rule,

$$\begin{aligned}&\frac{d}{dx}\left\{\begin{bmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right\}\\ =\ &\left(\frac{d}{dx}\begin{bmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} + \begin{bmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{bmatrix}\frac{d}{dx}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\end{aligned}$$

Thus the vector function

$$\begin{bmatrix}y\\ y'\end{bmatrix} = \begin{bmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

satisfies the system

$$\frac{d}{dx}\begin{bmatrix}y\\ y'\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}y\\ y'\end{bmatrix} + \begin{bmatrix}0\\ g\end{bmatrix}$$

if and only if

$$\begin{bmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}' = \begin{bmatrix}0\\ g\end{bmatrix}.$$

But $\det\begin{bmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{bmatrix} = \mathcal{W}(\varphi_1,\varphi_2)$ is nonzero, thus the matrix $\begin{bmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{bmatrix}$ is invertible, and so we can solve for $\begin{bmatrix}v_1\\ v_2\end{bmatrix}'$:

$$\begin{bmatrix}v_1\\ v_2\end{bmatrix}' = \begin{bmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{bmatrix}^{-1}\begin{bmatrix}0\\ g\end{bmatrix}.$$

Now we use the familiar formula for inverting a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

to obtain

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}' = \frac{1}{\mathcal{W}(\varphi_1,\varphi_2)} \begin{bmatrix} \varphi_2' & -\varphi_2 \\ -\varphi_1' & \varphi_1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} = \frac{1}{\mathcal{W}(\varphi_1,\varphi_2)} \begin{bmatrix} -\varphi_2 g \\ \varphi_1 g \end{bmatrix},$$

and hence finally

$$\begin{cases} v_1(x) &= -\int \frac{1}{\mathcal{W}(\varphi_1,\varphi_2)(x)} \varphi_2(x)\, g(x)\, dx \\ v_2(x) &= \int \frac{1}{\mathcal{W}(\varphi_1,\varphi_2)(x)} \varphi_1(x)\, g(x)\, dx \end{cases}.$$

Thus a particular solution y_p to the nonhomogeneous equation $L[y] = g$ is given by the first component of $\begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, which is

$$\begin{aligned} y_p(x) &= \varphi_1(x)\, v_1(x) + \varphi_2(x)\, v_2(x) \\ &= -\varphi_1(x) \int^x \frac{1}{\mathcal{W}(\varphi_1,\varphi_2)(s)} \varphi_2(s)\, g(s)\, ds \\ &\quad +\varphi_2(x) \int^x \frac{1}{\mathcal{W}(\varphi_1,\varphi_2)(s)} g(s)\, ds \\ &= \int^x \frac{1}{\mathcal{W}(\varphi_1,\varphi_2)(s)} \{\varphi_1(s)\,\varphi_2(x) - \varphi_1(x)\,\varphi_2(s)\}\, g(s)\, ds \\ &= \int^x \frac{1}{\mathcal{W}(\varphi_1,\varphi_2)(s)} \det \begin{bmatrix} \varphi_1(s) & \varphi_2(s) \\ \varphi_1(x) & \varphi_2(x) \end{bmatrix} g(s)\, ds, \end{aligned} \tag{4.1}$$

or written out in full,

$$y_p(x) = \int^x \frac{\det \begin{bmatrix} \varphi_1(s) & \varphi_2(s) \\ \varphi_1(x) & \varphi_2(x) \end{bmatrix}}{\det \begin{bmatrix} \varphi_1(s) & \varphi_2(s) \\ \varphi_1'(s) & \varphi_2'(s) \end{bmatrix}} g(s)\, ds. \tag{4.2}$$

REMARK 4. *We can use Abel's formula*

$$\mathcal{W}(\varphi_1,\varphi_2)(s) = e^{-\int_{s_0}^{s} a_1} \mathcal{W}(\varphi_1,\varphi_2)(s_0)$$

to calculate the Wronskian, but we must be careful when calculating a particular solution in formula (4.1), to use the exact *value of the Wronskian, and not just an arbitrary multiple of the function* $e^{-\int^s a_1}$.

EXAMPLE 26. *To find a particular solution to the equation*

$$y'' + 4y = 3\csc x,$$

we compute the players in formula (4.2) above. We start with a fundamental solution set for $y'' + 4y = 0$*, for which we can take*

$$\{\varphi_1(x), \varphi_2(x)\} = \{\cos 2x, \sin 2x\},$$

and then compute the determinants

$$\begin{aligned} \det \begin{bmatrix} \varphi_1(s) & \varphi_2(s) \\ \varphi_1(x) & \varphi_2(x) \end{bmatrix} &= \det \begin{bmatrix} \cos 2s & \sin 2s \\ \cos 2x & \sin 2x \end{bmatrix} \\ &= \cos 2s \sin 2x - \cos 2x \sin 2s, \end{aligned}$$

and

$$\begin{aligned}
\det\begin{bmatrix} \varphi_1(s) & \varphi_2(s) \\ \varphi_1'(s) & \varphi_2'(s) \end{bmatrix} &= \det\begin{bmatrix} \cos 2s & \sin 2s \\ -2\sin 2s & 2\cos 2s \end{bmatrix} \\
&= 2\cos^2 2s + 2\sin^2 2s = 2.
\end{aligned}$$

Thus (4.2), and the double angle formulas $\cos 2s = 1 - 2\sin^2 s$ *and* $\sin 2s = 2\sin s\cos s$, *give*

$$\begin{aligned}
y_p(x) &= \int^x \frac{\cos 2s \sin 2x - \cos 2x \sin 2s}{2} 3\csc s ds \\
&= \frac{3}{2}\int^x \frac{\left(1-2\sin^2 s\right)\sin 2x - \cos 2x\,(2\sin s\cos s)}{\sin s} ds \\
&= \frac{3}{2}\sin 2x \int^x (\csc s - 2\sin s)\, ds - \frac{3}{2}\cos 2x \int^x 2\cos s ds \\
&= \frac{3}{2}\sin 2x \left\{\ln|\csc x - \cot x| + 2\cos x\right\} - 3\cos 2x \sin x \\
&= \frac{3}{2}\sin 2x \ln|\csc x - \cot x| + 3\sin x,
\end{aligned}$$

since $\sin 2x\cos x - \cos 2x \sin x = \sin(2x - x)$.

5. Cauchy-Euler equations

There is one very special type of n^{th} order equation with variable coefficients that we can solve explicitly, namely the Cauchy-Euler equation

$$L[y] \equiv a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + ... + a_1 x y' + a_0 y = g, \tag{5.1}$$

where the linear operator $L[y]$ has coefficients $(a_k x^k)$ that are *constant* (a_k) multiples of monomials (x^k) whose degree matches the order (k) of the derivative $y^{(k)}$. This equation has a singular point at $x = 0$, and so from the existence and uniqueness theorems, we can only expect solutions to be defined on the half lines $(0, \infty)$ and $(-\infty, 0)$, and not at $x = 0$. We will confine our attention to $x > 0$, the case $x < 0$ being similar.

The substitution $x = e^t$ for $x > 0$ turns out to reduce the Cauchy-Euler equation (5.1) to a *constant* coefficient equation, that we can then solve using the techniques in the previous sections. Indeed, using the chain rule

$$\begin{aligned}
\frac{dx}{dt} &= e^t = x, \\
\frac{dt}{dx} &= \frac{1}{\frac{dx}{dt}} = \frac{1}{x}, \\
\frac{d}{dx} &= \frac{dt}{dx}\frac{d}{dt} = \frac{1}{x}\frac{d}{dt},
\end{aligned}$$

repeatedly then gives

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}y = \frac{1}{x}\frac{d}{dt}y = \frac{1}{x}\frac{dy}{dt};\\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} = \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)\\
&= \frac{1}{x}\left\{-\frac{1}{x^2}x\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\right\} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right);\\
\frac{d^3y}{dx^3} &= \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right)\\
&= \frac{1}{x}\left\{-\frac{2}{x^3}x\frac{d^2y}{dt^2} + \frac{1}{x^2}\frac{d^3y}{dt^3} + 2\frac{1}{x^3}x\frac{dy}{dt} - \frac{1}{x^2}\frac{d^2y}{dt^2}\right\}\\
&= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right);\\
&\quad \vdots
\end{aligned}$$

and

$$\begin{aligned}
a_1xy' &= a_1\frac{dy}{dt} + a_0y;\\
a_2x^2y'' + a_1xy' + a_0y &= a_2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1\frac{dy}{dt} + a_0y\\
&= a_2\frac{d^2y}{dt^2} + (a_1 - a_2)\frac{dy}{dt} + a_0y;\\
a_3x^3y''' + a_2x^2y'' + a_1xy' + a_0y &= a_3\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)\\
&\quad + a_2\frac{d^2y}{dt^2} + (a_1 - a_2)\frac{dy}{dt} + a_0y\\
&= a_3\frac{d^3y}{dt^3} + (a_2 - 3a_3)\frac{d^2y}{dt^2} + (a_1 - a_2 + 2a_3)\frac{dy}{dt} + a_0y;\\
&\quad \vdots
\end{aligned} \tag{5.2}$$

and so by induction we see that

$$L[y] = b_n\frac{d^ny}{dt^n} + b_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ...b_1\frac{dy}{dt} + b_0y,$$

where $b_0, b_2, ...b_n$ are certain constant coefficients depending on the a_k's. Now the first and last coefficients are easily identified as $b_0 = a_0$ and $b_n = a_n$, but the formulas for the intermediate coefficients b_k are not evident at this point. It turns out however that there is an *easy* method for computing these constants b_k that we will discover in a moment, so we leave them alone for now.

By Theorem 8 a fundamental solution set for equation (5.1) can be written down once we have factored the characteristic polynomial of the constant coefficient operator,

$$\begin{aligned}
P(r) &= b_n r^n + b_{n-1} r^{n-1} + ... b_1 r + b_0 \\
&= (r - r_1)^{k_1} (r - r_2)^{k_2} ... (r - r_M)^{k_M} \\
&\quad \times \{(r - [\alpha_1 + i\beta_1])(r - [\alpha_1 - i\beta_1])\}^{\ell_1} ... \{(r - [\alpha_N + i\beta_N])(r\lambda - [\alpha_N - ib\beta_N])\}^{\ell_N},
\end{aligned}$$

where the r_j, α_j, β_j are all real numbers, and

$$n = k_1 + ... + k_M + 2\ell_1 + ... + 2\ell_N.$$

We replace all the x's in the fundamental solution set (3.3) by t's, and then use our substitution $x = e^t$ to plug in the identities

$$\begin{aligned}
e^{rt} &= x^r \text{ and } t^k e^{rt} = (\ln x)^k x^r, \\
\cos \beta t &= \cos(\beta \ln x) \text{ and } \sin \beta t = \sin(\beta \ln x).
\end{aligned}$$

The result is that the set of n functions

$$\begin{aligned}
&x^{r_1},\ (\ln x) x^{r_1}, ... (\ln x)^{k_1 - 1} x^{r_1}; \\
&x^{r_2},\ (\ln x) x^{r_2}, ... (\ln x)^{k_2 - 1} x^{r_2}; \\
&\vdots \\
&x^{r_M},\ (\ln x) x^{r_M}, ... (\ln x)^{k_M - 1} x^{r_M}; \\
&x^{\alpha_1} \cos(\beta_1 \ln x),\ (\ln x) x^{\alpha_1} \cos(\beta_1 \ln x), ... (\ln x)^{k_1 - 1} x^{\alpha_1} \cos(\beta_1 \ln x); \\
&x^{\alpha_1} \sin(\beta_1 \ln x),\ (\ln x) x^{\alpha_1} \sin(\beta_1 \ln x), ... (\ln x)^{k_1 - 1} x^{\alpha_1} \sin(\beta_1 \ln x); \\
&\vdots \\
&x^{\alpha_N} \cos(\beta_N \ln x),\ (\ln x) x^{\alpha_N} \cos(\beta_N \ln x), ... (\ln x)^{\ell_N - 1} x^{\alpha_N} \cos(\beta_{N1} \ln x); \\
&x^{\alpha_N} \sin(\beta_N \ln x),\ (\ln x) x^{\alpha_N} \sin(\beta_N \ln x), ... (\ln x)^{\ell_N - 1} x^{\alpha_N} \sin(\beta_N \ln x),
\end{aligned} \tag{5.3}$$

is a fundamental solution set for the homogeneous Cauchy-Euler equation $L[y] = 0$.

Conclusion 3. *A fundamental solution set for the Cauchy-Euler equation is obtained from a fundamental solution set for the corresponding constant coefficient equation, by replacing x with $\ln x$ everywhere.*

In order to minimize confusion, we will refer to the characteristic polynomial $P(r)$ of the associated constant coefficient equation, as the *indicial polynomial* $P(\lambda)$ associated with the Cauchy-Euler equation (5.1), and write the variable as λ instead of r to help us remember this. In some books the equation $P(\lambda) = 0$ is called the auxilliary equation. The above conclusion then suggests that in order to determine the indicial polynomial $P(\lambda)$ associated with the Cauchy-Euler equation (5.1), we can just substitute the function x^λ into the Cauchy-Euler equation (5.1):

$$\begin{aligned}
L\left[x^\lambda\right] &= a_n x^n \left[\lambda(\lambda - 1) ... (\lambda - (n-1)) x^{\lambda - n}\right] \\
&\quad + a_{n-1} x^{n-1} \left[\lambda(\lambda - 1) ... (\lambda - (n-2)) x^{\lambda - (n-1)}\right] + ... + a_1 x \lambda x^{\lambda - 1} + a_0 x^\lambda \\
&= x^\lambda \{a_n \lambda(\lambda - 1) ... (\lambda - (n-1)) + a_{n-1} \lambda(\lambda - 1) ... (\lambda - (n-2)) + ... a_1 \lambda + a_0\} \\
&= x^\lambda P(\lambda).
\end{aligned}$$

Thus $L\left[x^\lambda\right] = 0$ if and only if $P(\lambda) = 0$ and we have discovered that the indicial polynomial $P(\lambda)$ can be computed simply by plugging x^λ into the Cauchy-Euler equation and calculating derivatives. The case $n = 2$ is particularly simple:

$$P(\lambda) = a_2\lambda(\lambda - 1) + a_1\lambda + a_0,$$

which of course agrees with the characteristic polynomial $P(r) = a_2 r^2 + (a_1 - a_2) r + a_0$ of the associated constant coefficient equation, that is obtained from the third line in (5.2). We are now equipped to solve the Cauchy-Euler equation in three steps:

(1) Calculate the indicial polynomial $P(\lambda)$,
(2) Write down a fundamental solution set using (5.3),
(3) Solve the nonhomogeneous problem by the method of variation of parameters.

EXAMPLE 27. *To solve the nonhomogeneous equation*

$$x^2 y'' + 5xy' + 4y = x^{-1}, \qquad x > 0,$$

we compute

$$P(\lambda) = \lambda(\lambda - 1) + 5\lambda + 4 = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2,$$

so that $\left\{x^{-2}, (\ln x)\, x^{-2}\right\}$ *is a fundamental solution set for the homogeneous equation. To find a particular solution to the nonhomogeneous equation we set*

$$y_p = v_1 x^{-2} + v_2 (\ln x)\, x^{-2},$$

where

$$\mathbf{v}(x) = \int \mathbf{\Phi}(x)^{-1} \mathbf{g}(x)\, dx,$$

with $\mathbf{g}(x) = \begin{bmatrix} 0 \\ x^{-3} \end{bmatrix}$ *(remember to obtain g from the normalized equation!) and*

$$\begin{aligned}
\mathbf{\Phi}(x) &= \begin{bmatrix} x^{-2} & (\ln x)\, x^{-2} \\ -2x^{-3} & x^{-3}(1 - 2(\ln x)) \end{bmatrix}; \\
\mathbf{\Phi}(x)^{-1} &= \frac{1}{x^{-5}} \begin{bmatrix} x^{-3}(1 - 2(\ln x)) & -(\ln x)\, x^{-2} \\ 2x^{-3} & x^{-2} \end{bmatrix} = \begin{bmatrix} x^2(1 - 2(\ln x)) & -(\ln x)\, x^3 \\ 2x^2 & x^3 \end{bmatrix}; \\
\mathbf{\Phi}(x)^{-1} \mathbf{g}(x) &= \begin{bmatrix} x^2(1 - 2\ln x) & -(\ln x)\, x^3 \\ 2x^2 & x^3 \end{bmatrix} \begin{bmatrix} 0 \\ x^{-3} \end{bmatrix} = \begin{bmatrix} -\ln x \\ 1 \end{bmatrix}.
\end{aligned}$$

Thus we have

$$\begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix} = \int \mathbf{\Phi}(x)^{-1} \mathbf{g}(x)\, dx = \int \begin{bmatrix} -\ln x \\ 1 \end{bmatrix} dx = \begin{bmatrix} -x\ln x + x \\ x \end{bmatrix},$$

and so the top component of

$$\mathbf{\Phi}(x)\, \mathbf{v}(x) = \begin{bmatrix} x^{-2} & (\ln x)\, x^{-2} \\ -2x^{-3} & x^{-3}(1 - 2(\ln x)) \end{bmatrix} \begin{bmatrix} -x\ln x + x \\ x \end{bmatrix} = \begin{bmatrix} x^{-1} \\ *** \end{bmatrix}$$

is a particular solution, i.e. $y_p = x^{-1}$. *Then the general solution is given by*

$$y(x) = y_p(x) + y_c(x) = x^{-1} + c_1 x^{-2} + c_2 (\ln x)\, x^{-2}, \qquad x > 0.$$

REMARK 5. *It is also possible in this example to use the method of undetermined coefficients, for the corresponding constant coefficient equation, to see that a particular solution* y_p *has the form* Ax^{-1}, *but in general we must use variation of parameters.*

EXAMPLE 28. *To solve the homogeneous equation*

$$x^2y'' + xy' + 9y = 0,$$

we compute

$$P(\lambda) = \lambda(\lambda - 1) + \lambda + 9 = \lambda^2 + 9 = (\lambda + 3i)(\lambda - 3i),$$

so that $\{\cos(3\ln x), \sin(3\ln x)\}$ *is a fundamental solution set. The general solution is given by*

$$y(x) = y_c(x) = c_1\cos(3\ln x) + c_2\sin(3\ln x), \qquad x > 0.$$

CHAPTER 5

Power series solutions

In the previous chapter we showed how to successfully solve the n^{th} order linear equation (2.1),

$$L[y] = a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + ... + a_1(x) y' + a_0(x) y = g(x),$$

in the following special cases:

- when the coefficients $a_k(x) = a_k$ are constants, and the forcing function $g(x) \equiv 0$ vanishes, we can find a fundamental solution set $\Phi = \{\varphi_1, \varphi_2, ...\varphi_n\}$;
- when the coefficients $a_k(x) = a_k$ are constants, and the forcing function $g(x)$ has a special form, we can use undetermined coefficients to find a particular solution y_p, and hence the general solution as well;
- when we have a fundamental solution set Φ, even when the coefficients are variable, we can use variation of parameters to find the general solution;
- when $n = 2$, and we know a nontrivial solution to the homogeneous equation $L[y] = 0$, we can use the method of reduction of order to find the general solution. Abel's formula can be used instead if we only want to find a second independent solution to the homogeneous equation.

Thus in the absence of any special information regarding solutions, the only case we can always solve so far is the case when the coefficients $a_k(x) = a_k$ are constant. In the general case of variable coefficients, we cannot find closed forms for the solutions, despite the fact that their existence is guaranteed by the Existence and Uniqueness Theorems. But for a very large class of equations with 'nice' variable coefficients, we can always find solutions in the form of power series, i.e. solutions $y(x)$ having the form

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n (x - x_0)^n \\ &= c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + ... \\ \text{for } -R &< x - x_0 < R, \end{aligned} \tag{0.4}$$

where

(1) x_0 is a real number called the *center of the power series* expansion, and
(2) the c_n are real numbers called the *power series coefficients*, and
(3) R is a positive real number, perhaps infinity, called the *radius of convergence* of the power series, such that the series $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ converges for all $x \in (x_0 - R, x_0 + R)$, called the *interval of convergence* of the power

series, i.e.

$$\lim_{N\to\infty}\sum_{n=0}^{N} c_n\left(x-x_0\right)^n \text{ exists for all } -R<x-x_0<R.$$

A function $y(x)$, that is defined by a power series as above, is said to be *analytic* at the point x_0. To be precise, $y(x)$ is analytic at a point x_0 if it is defined in some nontrivial open interval (x_0-R, x_0+R) centered at x_0, and is given by a convergent power series as in (0.4). The theory of power series is covered in detail in virtually every first year calculus book, and in particular, in the book by James Stewart, to which we refer the reader. Before describing the theory surrounding power series solutions to linear differential equations, we apply the method to Airy's equation in order to focus our thoughts:

$$y'' - xy = 0, \qquad -\infty < x < \infty. \tag{0.5}$$

Here we hope to solve the equation with power series of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n, \qquad -\infty < x < \infty, \tag{0.6}$$

in which the series is centered at $x_0 = 0$, and the radius of convergence is $R = \infty$ infinite. The method we use is often called the method of undetermined *series* coefficients, and in analogy with the method of undetermined coefficients discussed in the previous chapter, we simply plug the guessed form (0.6) of the solution into the equation and solve for the coefficients c_n. So we need to know how to compute both y'' and xy when y is a power series.

Since a power series can be differentiated term by term within its interval of convergence, we have

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left\{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\right\} \\
&= \frac{d}{dx}c_0 + \frac{d}{dx}c_1 x + \frac{d}{dx}c_2 x^2 + \frac{d}{dx}c_3 x^3 + \ldots \\
&= 0 + c_1 + c_2 2x + c_3 3x^2 + \ldots \\
&= c_1 + 2c_2 x + 3c_3 x^2 + \ldots,
\end{aligned}$$

equivalently

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left\{\sum_{n=0}^{\infty} c_n x^n\right\} \\
&= \sum_{n=0}^{\infty} \frac{d}{dx} c_n x^n \\
&= \sum_{n=0}^{\infty} c_n n x^{n-1} \\
&= \sum_{n=1}^{\infty} n c_n x^{n-1},
\end{aligned}$$

and by another application of term by term differentiation,

$$\begin{aligned} y''(x) &= \frac{d}{dx}\left\{c_1 + 2c_2x + 3c_3x^2 + ...\right\} \\ &= \frac{d}{dx}c_1 + \frac{d}{dx}2c_2x + \frac{d}{dx}3c_3x^2 + ... \\ &= 0 + 2c_2 + c_3 6x + ... \\ &= 2c_2 + 6c_3x + ..., \end{aligned}$$

equivalently

$$\begin{aligned} y''(x) &= \frac{d}{dx}y'(x) = \frac{d}{dx}\left\{\sum_{n=1}^{\infty} nc_nx^{n-1}\right\} \\ &= \sum_{n=1}^{\infty}\frac{d}{dx}c_nnx^{n-1} \\ &= \sum_{n=1}^{\infty}c_nn(n-1)x^{n-2} \\ &= \sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}. \end{aligned}$$

By the rule for multiplying series we have

$$\begin{aligned} xy &= x\left\{c_0 + c_1x + c_2x^2 + c_3x^3 + ...\right\} \\ &= c_0x + c_1x^2 + c_2x^3 + c_3x^4 + ... \end{aligned}$$

equivalently

$$xy = x\sum_{n=0}^{\infty}c_nx^n = \sum_{n=0}^{\infty}xc_nx^n = \sum_{n=0}^{\infty}c_nx^{n+1}.$$

If we now substitute these two expressions into Airy's equation we get

$$\begin{aligned} 0 &= 2c_2 + 6c_3x + 24c_4x^2 + ... \\ &\quad -c_0x + c_1x^2 + c_2x^3 + c_3x^4 + ... \\ &= 2c_2 + (6c_3 - c_0)x + (24c_4 - c_1)x^2 + ... \end{aligned}$$

equivalently

$$0 = y'' - xy = \sum_{n=2}^{\infty}n(n-1)c_nx^{n-2} - \sum_{n=0}^{\infty}c_nx^{n+1}.$$

Now the first form, in which we write out the initial terms, is useful for 'seeing' what is going on, and by equating coefficients of like powers of x (all the coefficients on the left side vanish) we see that $c_2 = 0$, $c_3 = \frac{1}{6}c_0$, and $c_4 = \frac{1}{24}c_1$; but this form is not sufficiently explicit to help us easily solve for the remainder of the coefficients c_n. Instead we would like to use the second form involving the infinite summation notation $\sum$. But a difficulty here is that the powers of x don't match in the two series, namely the series for y'' has x^{n-2} while the series for xy has x^{n+1}. We remedy this situation by *shifting the index of summation* so as to have x^n appear in both series.

In the series for y'', this can be accomplished by making the substitution $n = k+2$,

$$\begin{aligned} y'' &= \sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} \\ &= \sum_{k+2=2}^{\infty} (k+2)(k+2-1)\, c_{k+2} x^{k+2-2} \\ &= \sum_{k=0}^{\infty} (k+2)(k+1)\, c_{k+2} x^{k} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^{n}, \end{aligned}$$

where in the last line we have replaced the dummy index k by the dummy index n. Once familiar with the process of shifting the index of summation, one can simply replace n with $n+2$ here, and skip the introduction of the auxilliary dummy index k. The skeptical reader should write out the initial terms in the above lines to see what is going on here.

In the series for xy this is accomplished by the substitution $n = k-1$,

$$\begin{aligned} xy &= \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= \sum_{k-1=0}^{\infty} c_{k-1} x^{k-1+1} \\ &= \sum_{k=1}^{\infty} c_{k-1} x^{k} \\ &= \sum_{n=1}^{\infty} c_{n-1} x^{n}. \end{aligned}$$

Now the two series have the same power x^n appearing, and they only differ in where the summation starts, namely the shifted series for y'' now starts at $n = 0$, but the shifted series for xy starts at $n = 1$. This means that we have to separate out the case $n = 0$ and write

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^{n} - \sum_{n=1}^{\infty} c_{n-1} x^{n} \\ &= (0+2)(0+1)\, c_{0+2} x^{0} + \sum_{n=1}^{\infty} \left\{ (n+2)(n+1)\, c_{n+2} x^{n} - c_{n-1} x^{n} \right\} \\ &= 2c_2 + \sum_{n=1}^{\infty} \left\{ (n+2)(n+1)\, c_{n+2} - c_{n-1} \right\} x^{n}. \end{aligned}$$

Next, we equate coefficients of like powers of x on each side - and all the coefficients on the left side are 0 - to obtain

$$\begin{aligned} 0 &= 2c_2, \\ 0 &= 6c_3 - c_0, \\ 0 &= 12c_4 - c_1, \\ 0 &= 20c_5 - c_2, \\ &\vdots \\ 0 &= (n+2)(n+1)\, c_{n+2} - c_{n-1}, \\ &\vdots \end{aligned}$$

which we can write as a *recursion relation*,

$$\begin{aligned} c_2 &= 0, \\ c_{n+2} &= \frac{1}{(n+2)(n+1)} c_{n-1}, \qquad n \geq 1. \end{aligned} \tag{0.7}$$

This specifies all of the coefficients c_n for $n \geq 2$ in terms of the first two coefficients c_0 and c_1, which are left unrestricted. Thus we see that for every choice of constants c_0 and c_1, we obtain a *possible* power series solution to Airy's equation having the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$, where the coefficients c_n satisfy the recurrence relation (0.7). We don't yet know these series are actually solutions since we don't even know at this point if the series converge anywhere other than at $x = 0$.

This particular recurrence relation (0.7) can be explicitly solved by induction as follows. In order to start with c_n, we make the substitution $n \to n-2$ in the relation to get

$$c_n = \frac{1}{n(n-1)} c_{n-3}, \qquad n \geq 3,$$

and then, since c_{n-3} appears on the right side, we make the further substitution $n \to n-3$ to get

$$c_{n-3} = \frac{1}{(n-3)(n-4)} c_{n-6}, \qquad n \geq 6,$$

so that we have

$$\begin{aligned} c_n &= \frac{1}{n(n-1)} c_{n-3} \\ &= \frac{1}{n(n-1)} \frac{1}{(n-3)(n-4)} c_{n-6}. \end{aligned}$$

Continuing in this manner, for $n = 3k$ we get by induction on k,

$$\begin{aligned} c_{3k} &= \frac{1}{3k(3k-1)} c_{3(k-1)} \\ &= \frac{1}{3k(3k-1)} \frac{1}{(3k-3)(3k-4)} c_{3(k-2)} \\ &\vdots \\ &= \frac{1}{3k(3k-1)} \frac{1}{(3k-3)(3k-4)} \cdots \frac{1}{(3)(2)} c_0, \qquad k \geq 0, \end{aligned}$$

where it is understood that when $k = 0$, the empty product is 1. Similarly we have for $n = 3k + 1$,

$$c_{3k+1} = \frac{1}{(3k+1)(3k)} \frac{1}{(3k-2)(3k-3)} \cdots \frac{1}{(4)(3)} c_1, \qquad k \geq 0,$$

where again, when $k = 0$, the empty product is 1. Finally

$$c_{3k+2} = \frac{1}{(3k+2)(3k+1)} \cdots \frac{1}{(5)(4)} c_2 = 0, \qquad k \geq 0.$$

Thus we have now constructed the following possible series solutions where we separate out the sums over $n = 3k$, $3k + 1$ and $3k + 2$:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{3k} x^{3k} + \sum_{k=0}^{\infty} c_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} c_{3k+2} x^{3k+2} \\ &= c_0 \sum_{k=0}^{\infty} \left\{ \frac{1}{3k(3k-1)} \frac{1}{(3k-3)(3k-4)} \cdots \frac{1}{(3)(2)} \right\} x^{3k} \\ &\quad + c_1 \sum_{k=0}^{\infty} \left\{ \frac{1}{(3k+1)(3k)} \frac{1}{(3k-2)(3k-3)} \cdots \frac{1}{(4)(3)} \right\} x^{3k+1} \\ &\quad + \sum_{k=0}^{\infty} 0 \cdot x^{3k+2} \\ &= c_0 y_0(x) + c_1 y_1(x), \end{aligned}$$

where

$$\begin{aligned} y_0(x) &= \sum_{k=0}^{\infty} \left\{ \frac{1}{3k(3k-1)} \frac{1}{(3k-3)(3k-4)} \cdots \frac{1}{(3)(2)} \right\} x^{3k} \equiv \sum_{k=0}^{\infty} b_{3k} x^{3k}, \\ y_1(x) &= \sum_{k=0}^{\infty} \left\{ \frac{1}{(3k+1)(3k)} \frac{1}{(3k-2)(3k-3)} \cdots \frac{1}{(4)(3)} \right\} x^{3k+1} \equiv \sum_{k=0}^{\infty} b_{3k+1} x^{3k+1}. \end{aligned}$$

It remains to see if these possible series solutions actually converge for all $x \in (-\infty, \infty)$, and moreover to a solution to Airy's equation. For this we recall the ratio test for series.

LEMMA 4 (Ratio Test). *A series* $\sum_{n=0}^{\infty} a_n$ *converges absolutely if*

$$L \equiv \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

and diverges if $L > 1$.

To apply the Ratio Test to the power series $y_0(x) = \sum_{k=0}^{\infty} b_{3k} x^{3k}$, we write

$$a_k \equiv b_{3k} x^{3k}$$

and compute

$$\begin{aligned} L &= \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{b_{3(k+1)}x^{3(k+1)}}{b_{3k}x^{3k}}\right| = |x|^3\lim_{k\to\infty}\left|\frac{b_{3(k+1)}}{b_{3k}}\right| \\ &= |x|^3\lim_{k\to\infty}\left|\frac{\frac{1}{3(k+1)(3(k+1)-1)}\frac{1}{(3(k+1)-3)(3(k+1)-4)}\cdots\frac{1}{(3)(2)}}{\frac{1}{3k(3k-1)}\frac{1}{(3k-3)(3k-4)}\cdots\frac{1}{(3)(2)}}\right| \\ &= |x|^3\lim_{k\to\infty}\left|\frac{\frac{1}{3(k+1)(3(k+1)-1)}}{1}\right| = 0 < 1. \end{aligned}$$

It follows from the Ratio Test that the series for $y_0(x)$ converges for all $x \in (-\infty,\infty)$. A similar calculation shows that the series for $y_1(x)$ also converges for all $x \in (-\infty,\infty)$.

Then we can show that both $y_0(x)$ and $y_1(x)$ are solutions to Airy's equation (0.5), by using the facts that term by term differentiation and multiplication of power series are valid within their open intervals of convergence. Finally, from

$$\begin{aligned} y_0(0) &= b_0 = 1, \qquad y_0'(0) = 0, \\ y_1(0) &= b_1 = 0, \qquad y_1'(0) = b_1 = 1, \end{aligned}$$

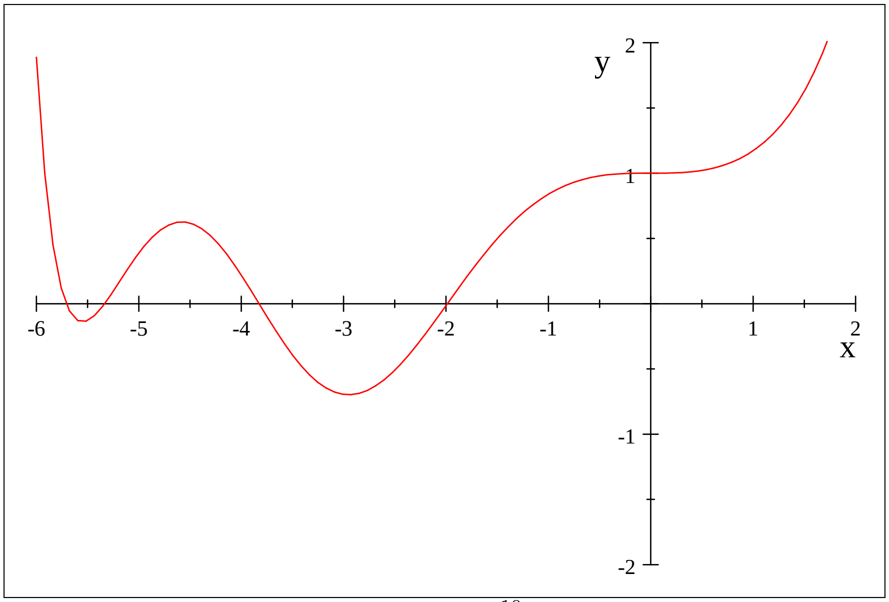
we see that the solutions $y_0(x)$ and $y_1(x)$ are *linearly independent* since their Wronskian satisfies

$$\mathcal{W}(y_0,y_1)(0) = \det\begin{bmatrix} y_0(0) & y_1(0) \\ y_0'(0) & y_1'(0)\end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} = 1 \neq 0.$$

Thus $\{y_0,y_1\}$ is a fundamental solution set, and the general solution of Airy's equation (0.5) is

$$y(x) = c_0y_0(x) + c_1y_1(x), \qquad x \in (-\infty,\infty).$$

REMARK 6. *For $x < 0$, we might expect that the Airy solutions $y_0(x)$ and $y_1(x)$ behave qualitatively like the solutions to $y'' + y = 0$, namely $\cos x$ and $\sin x$. Similarly, for $x > 0$, we might expect $y_0(x)$ and $y_1(x)$ to behave qualitatively like the solutions to $y'' - y = 0$, namely $\cosh x$ and $\sinh x$. That this is roughly so can be seen from the graphs of partial sums of the power series for $y_0(x)$ and $y_1(x)$. Pictured below is the graph of the partial sum for $y_0(x)$ of degree 30. It is a reasonably accurate approximation to $y_0(x)$ in the range $-5.5 < x < 2$, but for $x < -6$, the graph of $y_0(x)$ continues to oscillate like $\cos x$.*



The graph of $y = \sum_{k=0}^{10} b_{3k} x^{3k}$.

1. Theory of power series solutions

We consider the problem of finding a fundamental solution set for the n^{th} order homogeneous linear equation

$$L[y] = a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots + a_1(x)\, y' + a_0(x)\, y = 0, \tag{1.1}$$

where the variable coefficients $a_k(x)$ are all analytic at some point x_0. Once we have a fundamental solution set to the homogeneous equation (1.1), we can use the method of variation of parameters to solve the associated nonhomogeneous equations, and so we work only with the homogeneous equation in this chaper. Recall the definition of analytic:

Definition 3. *A function $a(x)$ is analytic at a point x_0 if it is defined in a nontrivial open interval $(x_0 - R, x_0 + R)$ in which it is given by a convergent power series centered at x_0:*

$$a(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad \text{for all } x \in (x_0 - R, x_0 + R).$$

A function $a(x)$ is analytic in an interval $I = (b, c)$ if it is analytic at each point $x_0 \in I$. Here the interval I can be finite or semi-infinite or the entire real line.

Power series solutions are most easily dealt with when the point x_0 is especially 'ordinary'.

Definition 4. *We say that x_0 is an* ordinary point *for the equation (1.1), if all the coefficients $a_k(x)$ are analytic at x_0, and if $a_n(x_0) \neq 0$. Otherwise we say x_0 is a* singular point *for the equation (1.1).*

If x_0 is an ordinary point for (1.1), we can write the equation (1.1) in normal form

$$L[y] = y^{(n)} + \frac{a_{n-1}(x)}{a_n(x)} y^{(n-1)} + \ldots + \frac{a_1(x)}{a_n(x)} y' + \frac{a_0(x)}{a_n(x)} y = 0,$$

where the normalized coefficients $\frac{a_k(x)}{a_n(x)}$ are all analytic at x_0. Then the Existence and Uniqueness Theorem applies to show that a fundamental solution set exists on any common interval of definition of the normalized coefficients. But we can actually do better in this case, and obtain a fundamental solution set consisting of analytic functions. But before stating the theorem, we point out that the definition of ordinary and singular points given above applies as well to *complex* points $x_0 \in \mathbb{C}$. This plays a role in determining the largest common radius of convergence for the functions in our fundamental solution set. Here is the main theorem regarding power series solutions about an ordinary point.

THEOREM 10. *Suppose x_0 is an ordinary point for the equation (1.1). Then there is a fundamental solution set $\{y_1, y_2, ..., y_n\}$ where each y_k has a power series expansion about x_0 with radius of convergence $R > 0$:*

$$y_k(x) = \sum_{m=0}^{\infty} c_{k,m} (x - x_0)^m, \qquad x \in (x_0 - R, x_0 + R), 1 \le k \le n,$$

and where R is the distance from x_0 to the nearest singular point in the complex plane $\mathbb{C}$. Moreover, the coefficients $c_{k,m}$ can be determined by substitution in the equation (1.1).

1.1. Equivalence with an $n \times n$ system. In order to prove Theorem 10, we again exploit the algebra of square matrices by considering the n^{th} order equation (1.1) as the $n \times n$ system (1.1):

$$\left\{ \begin{array}{ccc} y_1' & = & y_2 \\ y_2' & = & y_3 \\ \vdots & \vdots & \vdots \\ y_{n-1}' & = & y_n \\ y_n' & = & f(x, y_1, y_2, ..., y_{n-1}) \end{array} \right. ,$$

where f is given by the normalized form,

$$f(x, y_1, y_2, ..., y_{n-1}) = -\left\{ \frac{a_{n-1}(x)}{a_n(x)} y^{(n-1)} + ... + \frac{a_1(x)}{a_n(x)} y' + \frac{a_0(x)}{a_n(x)} y \right\},$$

and the coefficients $\frac{a_k(x)}{a_n(x)}$ are analytic at an ordinary point x_0. In fact we will prove the analogue of Theorem 10 for the *general* first order *linear* system of n equations in n unknown functions $\{y_1, y_2, ..., y_n\}$,

$$\left\{ \begin{array}{ccccc} y_1' & = & f_1(x, y_1, y_2, ..., y_{n-1}) & = & a_{1,1}(x) y_1 + ... a_{1,n}(x) y_n \\ y_2' & = & f_2(x, y_1, y_2, ..., y_{n-1}) & = & a_{2,1}(x) y_1 + ... a_{2,n}(x) y_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{n-1}' & = & f_{n-1}(x, y_1, y_2, ..., y_{n-1}) & = & a_{n-1,1}(x) y_1 + ... a_{n-1,n}(x) y_n \\ y_n' & = & f_n(x, y_1, y_2, ..., y_{n-1}) & = & a_{n,1}(x) y_1 + ... a_{n,n}(x) y_n \end{array} \right. .$$

If we define the $n \times n$ matrix-valued function

$$A(x) \equiv \begin{bmatrix} a_{1,1}(x) & a_{1,2}(x) & \cdots & a_{1,n}(x) \\ a_{2,1}(x) & a_{2,2}(x) & \cdots & a_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n}(x) \end{bmatrix},$$

and the solution vector

$$\mathbf{y}(x) \equiv \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

we can write the $n \times n$ system as

$$\mathbf{y}'(x) = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{bmatrix} a_{1,1}(x) & a_{1,2}(x) & \cdots & a_{1,n}(x) \\ a_{2,1}(x) & a_{2,2}(x) & \cdots & a_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n}(x) \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = A(x)\,\mathbf{y}(x).$$

Moreover, if we write a fundamental solution set (of column vector solutions)

$$\Phi \equiv \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n\} = \left\{ \begin{pmatrix} y_{1,1} \\ y_{2,1} \\ \vdots \\ y_{n,1} \end{pmatrix}, \begin{pmatrix} y_{1,2} \\ y_{2,2} \\ \vdots \\ y_{n,2} \end{pmatrix}, ..., \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ \vdots \\ y_{n,n} \end{pmatrix} \right\},$$

in the form of a matrix with columns $\mathbf{y}_k$,

$$\Phi(x) \equiv \begin{bmatrix} y_{1,1}(x) & y_{1,2}(x) & \cdots & y_{1,n}(x) \\ y_{2,1}(x) & y_{2,2}(x) & \cdots & y_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1}(x) & y_{n,2}(x) & \cdots & y_{n,n}(x) \end{bmatrix}$$

then the equation for the fundamental solution set Φ becomes the following matrix equation,

$$\begin{aligned} \Phi'(x) &= \begin{bmatrix} y_{1,1}'(x) & y_{1,2}'(x) & \cdots & y_{1,n}'(x) \\ y_{2,1}'(x) & y_{2,2}'(x) & \cdots & y_{2,n}'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1}'(x) & y_{n,2}'(x) & \cdots & y_{n,n}'(x) \end{bmatrix} \\ &= \begin{bmatrix} a_{1,1}(x) & a_{1,2}(x) & \cdots & a_{1,n}(x) \\ a_{2,1}(x) & a_{2,2}(x) & \cdots & a_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n}(x) \end{bmatrix} \begin{bmatrix} y_{1,1}(x) & y_{1,2}(x) & \cdots & y_{1,n}(x) \\ y_{2,1}(x) & y_{2,2}(x) & \cdots & y_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1}(x) & y_{n,2}(x) & \cdots & y_{n,n}(x) \end{bmatrix} \\ &= A(x)\,\Phi(x), \end{aligned}$$

together with the linear independence (of the columns) condition

$$\det \Phi(x) \neq 0.$$

We say that the fundamental solution set Φ is *normalized* at x_0 if the matrix $\Phi(x_0)$ is the identity matrix.

Here then is the power series solution theorem for $\kappa \times \kappa$ systems, where we are writing the order of the system as κ so as to permit the use of n as a dummy variable in the series. We will also sometimes use the letter φ instead of c to denote coefficients.

THEOREM 11. *Suppose the coefficients $a_{j,k}(x)$ in the $\kappa \times \kappa$ matrix-valued function $A(x)$ are all analytic at x_0, and with radius of convergence at least $\rho > 0$. Then there is a fundamental solution set $\Phi = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_\kappa\}$ for the equation $\Phi' = A\Phi$ normalized at x_0, i.e. a matrix solution $\Phi(x)$ to the initial value problem*

$$\begin{aligned} \Phi'(x) &= A(x)\Phi(x), \\ \Phi(x_0) &= I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \end{aligned} \tag{1.2}$$

where each component $y_{j,k}(x)$ of the matrix $\Phi(x)$ has a power series expansion about x_0 with radius of convergence $R \geq \rho$:

$$y_{j,k}(x) = \sum_{n=0}^{\infty} \varphi_{j,k,n}(x - x_0)^n, \qquad x \in (x_0 - R, x_0 + R), 1 \leq j, k \leq \kappa.$$

Moreover, the coefficients $\varphi_{j,k,n}$ in the power series can be determined by substitution in the equation (1.2).

We will write the collection of series $y_{j,k}(x) = \sum_{n=0}^{\infty} \varphi_{j,k,n}(x - x_0)^n$ in matrix form as

$$\begin{aligned} \Phi(x) &= [y_{j,k}(x)]_{j,k=1}^{\kappa} = \left[\sum_{n=0}^{\infty} \varphi_{j,k,n}(x - x_0)^n\right]_{j,k=1}^{\kappa} \\ &= \sum_{n=0}^{\infty} \left[\varphi_{j,k,n}\right]_{j,k=1}^{\kappa} (x - x_0)^n \equiv \sum_{n=0}^{\infty} \Phi_n (x - x_0)^n, \end{aligned}$$

where the matrix coefficients Φ_n are given by

$$\Phi_n \equiv \left[\varphi_{j,k,n}\right]_{j,k=1}^{\kappa}, \qquad n \geq 0.$$

Simliarly, if

$$a_{j,k}(x) = \sum_{n=0}^{\infty} a_{j,k,n}(x - x_0)^n, \qquad x \in (x_0 - \rho, x_0 + \rho), 1 \leq j, k \leq n$$

we write

$$A(x) = \sum_{n=0}^{\infty} A_n (x - x_0)^n,$$

where

$$A_n \equiv [a_{j,k,n}]_{j,k=1}^{\kappa}, \qquad n \geq 0.$$

1.2. The Root Test. Given a sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we say that an extended real number $L \in [-\infty, \infty]$ is a *subsequential limit* of $\{s_n\}_{n=1}^{\infty}$ if there is a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ with

$$\lim_{k \to \infty} s_{n_k} = L.$$

We then define the limit superior of a sequence $\{s_n\}_{n=1}^{\infty}$, denoted $\limsup_{n\to\infty} s_n$, to be the largest subsequential limit of $\{s_n\}_{n=1}^{\infty}$. Similarly, $\liminf_{n\to\infty} s_n$ is the smallest subsequential limit of $\{s_n\}_{n=1}^{\infty}$. It is a standard theorem that both a largest and a smallest subsequential limit always exist in the extended real numbers.

Example 29. *The sequence* $\left\{\frac{(-1)^n n}{n+1}\right\}_{n=1}^{\infty}$ *has two subsequential limits,* ± 1, *and*

$$\limsup_{n\to\infty} \left\{\frac{(-1)^n n}{n+1}\right\}_{n=1}^{\infty} = 1,$$
$$\liminf_{n\to\infty} \left\{\frac{(-1)^n n}{n+1}\right\}_{n=1}^{\infty} = -1.$$

Example 30. *The sequence*

$$\left\{-1, 0, 1, \quad -\frac{4}{2}, -\frac{3}{2}, -\frac{2}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \right.$$
$$\left. -\frac{16}{4}, -\frac{15}{4}, ..., -\frac{1}{4}, 0, \frac{1}{4}, ..., \frac{15}{4}, \frac{16}{4}, \quad -\frac{64}{8}, -\frac{63}{8}, ...etc\right\}$$

has every *extended real number as a subsequential limit, and its limit superior is* ∞, *and its limit inferior is* $-\infty$.

Here is the root test for series.

Lemma 5 (The Root Test). *Let* $L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. *Then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely if* $L < 1$, *and diverges if* $L > 1$.

Proof. Suppose $L < 1$ and fix any number R with $L < R < 1$. Since L is the largest subsequential limit of $\left\{|a_n|^{\frac{1}{n}}\right\}_{n=1}^{\infty}$, it follows from a standard theorem that there is N such that

$$|a_n|^{\frac{1}{n}} < R, \qquad \text{for all } n \geq N.$$

But then $|a_n| < R^n$ for $n \geq N$, where $\sum_{n=0}^{\infty} R^n = \frac{1}{1-R} < \infty$, and so the comparison test shows that $\sum_{n=0}^{\infty} |a_n| < \infty$. If on the other hand $L > 1$, fix any number R with $1 < R < L$. Since L is a subsequential limit of $\left\{|a_n|^{\frac{1}{n}}\right\}_{n=1}^{\infty}$, it follows that there are infinitely many n for which $|a_n|^{\frac{1}{n}} > R$, hence infinitely many n for which $|a_n| > R^n > 1$. Thus the n^{th} term a_n does *not* tend to 0 as $n \to \infty$, and so the series $\sum_{n=0}^{\infty} a_n$ diverges. □

We now have the following corollary for the radius of convergence of a power series.

Corollary 1. *Let* $L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. *Then the radius of convergence of the power series* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is* $R = \frac{1}{L}$, *where we interpret* $\frac{1}{0}$ *as* ∞, *and* $\frac{1}{\infty}$ *as* 0.

Proof. For any x let

$$L(x) \equiv \limsup_{n\to\infty} |a_n (x - x_0)^n|^{\frac{1}{n}} = |x - x_0| \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = |x - x_0| L.$$

By the Root Test, the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges if $|x - x_0| L = L(x) < 1$, and diverges if $|x - x_0| L = L(x) > 1$; i.e. converges if $|x - x_0| < \frac{1}{L} = R$, and diverges if $|x - x_0| > \frac{1}{L} = R$. Thus the radius of convergence of $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is R. □

Both the Root Test and its corollary for power series extend to *matrix-valued* series if we use the following definition for the absolute value (or norm) of a matrix $A = [a_{j,k}]_{j,k=1}^{n}$:

$$\|A\| \equiv \left\|[a_{j,k}]_{j,k=1}^{n}\right\| = \left\| \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \right\| = \sqrt{\sum_{j,k=1}^{n} (a_{j,k})^2}.$$

In particular we state without proof the corollary for matrix-valued power series.

PROPOSITION 1. *Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of square matrices and set*

$$L = \lim \sup_{n\to\infty} \|A_n\|^{\frac{1}{n}} .$$

Then the radius of convergence of the matrix-valued power series

$$\sum_{n=0}^{\infty} A_n (x - x_0)^n$$

is $R = \frac{1}{L}$, i.e. the series $\sum_{n=0}^{\infty} A_n (x - x_0)^n$ converges (absolutely) if $|x - x_0| < R$, and diverges if $|x - x_0| > R$.

1.3. Proof of the power series solution theorem. In our proof of Theorem 11, we first demonstrate just the existence of *some* positive radius of convergence R for the series $\Phi(x) = \sum_{n=0}^{\infty} \Phi_n (x - x_0)^n$, and defer the proof that $R \geq \rho$ to the final subsection, as it involves a tricky induction. Moreover, we first give the complete details of this assertion only in the case $n = 1$, when the matrices are 1×1, hence just numbers. But the proof we give uses only standard properties of real-valued series that extend readily to matrices and their norms, and in the next subsection we sketch how to prove the case $n > 1$. Finally we assume without loss of generality that $x_0 = 0$.

So the scalar initial value problem corresponding to (1.2) is

$$\left\{ \begin{array}{ccc} \varphi'(x), & = & a(x)\varphi(x) \\ \varphi(0) & = & 1 \end{array} \right. , \tag{1.3}$$

where we assume that $a(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $\rho > 0$. We begin by assuming there is a power series solution

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n x^n$$

and plug it into the equation to get

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)\varphi_{n+1}x^n &= \sum_{n=1}^{\infty} n\varphi_n x^{n-1} = \varphi'(x) \\ &= a(x)\varphi(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{\ell=0}^{\infty} \varphi_\ell x^\ell\right) \\ &= \sum_{k,\ell=0}^{\infty} a_k \varphi_\ell x^{k+\ell} = \sum_{n=0}^{\infty}\left(\sum_{k+\ell=n}^{\infty} a_k\varphi_\ell\right) x^n. \end{aligned}$$

Equating coefficients of like powers of x gives the recurrence relation

$$(n+1)\,\varphi_{n+1} = \sum_{k+\ell=n}^{\infty} a_k \varphi_\ell, \qquad n \geq 0,$$

which after sending $n \to n+1$, we can write as

$$\varphi_n = \frac{1}{n} \sum_{k+\ell=n-1}^{\infty} a_k \varphi_\ell, \qquad n \geq 1.$$

The initial condition $\varphi(0) = 1$ implies that $\varphi_0 = \varphi(0) = 1$.

If we solve for the first few coefficients we get

$$\begin{aligned}
\varphi_0 &= 1, \\
\varphi_1 &= \frac{1}{1} \sum_{k+\ell=1-1}^{\infty} a_k \varphi_\ell = a_0 \varphi_0 = a_0, \\
\varphi_2 &= \frac{1}{2} \sum_{k+\ell=2-1}^{\infty} a_k \varphi_\ell = \frac{1}{2}(a_0 \varphi_1 + a_1 \varphi_0) = \frac{1}{2}\left(a_0^2 + a_1\right), \\
\varphi_3 &= \frac{1}{3} \sum_{k+\ell=3-1}^{\infty} a_k \varphi_\ell = \frac{1}{3}(a_0 \varphi_2 + a_1 \varphi_1 + a_2 \varphi_0) \\
&\qquad\qquad = \frac{1}{6}\left(a_0^3 + a_0 a_1\right) + \frac{1}{3} a_1 a_0 + \frac{1}{3} a_2.
\end{aligned}$$

This general recursion is difficult to solve, but there is an estimate we can make on the size of the coefficients φ_n if we use the information we have on the sequence $\{a_k\}_{k=0}^{\infty}$, namely that

$$\limsup_{k \to \infty} |a_k|^{\frac{1}{k}} = \frac{1}{\rho} < \infty.$$

From this it follows in particular that the sequence $\left\{|a_k|^{\frac{1}{k}}\right\}_{k=0}^{\infty}$, and so also $\left\{|a_k|^{\frac{1}{k+1}}\right\}_{k=0}^{\infty}$, is bounded, say by $M < \infty$:

$$|a_k|^{\frac{1}{k+1}} \leq M \text{ equivalently } |a_k| \leq M^{k+1}, \qquad \text{for all } k \geq 0.$$

If we use this estimate in the formulas above we get,

$$\begin{aligned}
|\varphi_1| &= |a_0| \leq M, \\
|\varphi_2| &= \frac{1}{2}|a_0 \varphi_1 + a_1 \varphi_0| \leq \frac{1}{2}\left(MM + M^2\right) = M^2, \\
|\varphi_3| &= \frac{1}{3}|a_0 \varphi_2 + a_1 \varphi_1 + a_2 \varphi_0| \leq \frac{1}{3}\left(MM^2 + M^2 M + M^3\right) = M^3,
\end{aligned}$$

which suggests that in general we have

$$|\varphi_n| \leq M^n, \qquad n \geq 0. \tag{1.4}$$

And indeed, this can be proved by induction:

$$|\varphi_n| = \left|\frac{1}{n} \sum_{k+\ell=n-1}^{\infty} a_k \varphi_\ell\right| \leq \frac{1}{n} \sum_{k+\ell=n-1}^{\infty} |a_k \varphi_\ell| \leq \frac{1}{n} \sum_{k+\ell=n-1}^{\infty} M^{k+1} M^\ell = M^n,$$

where we have used the induction assumption $|\varphi_\ell| \leq M^\ell$ for ℓ strictly less than n, together with the fact that there are exactly n summands in the sum $\sum_{k+\ell=n-1}^{\infty}$ since both k and ℓ are nonnegative.

From the estimate (1.4) we obtain

$$\limsup_{n\to\infty} |\varphi_n|^{\frac{1}{n}} \leq M < \infty,$$

and hence from Corollary 1, that the radius of convergence R of the power series $\varphi(x) = \sum_{n=0}^{\infty} \varphi_n x^n$ satisfies $R \geq \frac{1}{M} > 0$. It now follows from standard theorems on power series that $\varphi(x)$ solves the intial value problem (1.3) for $|x| < R$.

1.4. The higher order case. The above proof generalizes easily to higher orders. Here is a sketch of the arguments adapted to $\kappa \times \kappa$ square matrices. If we plug the matrix-valued series $\Phi(x) = \sum_{n=0}^{\infty} \Phi_n x^n$ into the matrix equation (1.2) and use $A(x) = \sum_{n=0}^{\infty} A_n x^n$, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1) \Phi_{n+1} x^n &= \Phi'(x) = A(x) \Phi(x) \\
&= \left(\sum_{k=0}^{\infty} A_k x^k \right) \left(\sum_{\ell=0}^{\infty} \Phi_\ell x^\ell \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k+\ell=n} A_k \Phi_\ell \right) x^n,
\end{aligned}$$

which gives the following recursion relation for the matrix coefficients Φ_n:

$$\Phi_n = \frac{1}{n} \sum_{k+\ell=n-1} A_k \Phi_\ell. \tag{1.5}$$

Now the matrix-valued series $\sum_{k=0}^{\infty} A_k x^k$ converges for $|x| < \rho$ by assumption, and so the Root Test for matrix-valued series, Proposition 1, shows that $\limsup_{k\to\infty} \|A_k\|^{\frac{1}{k}} \leq \frac{1}{\rho}$, from which it follows that $\left\{ \|A_k\|^{\frac{1}{k}} \right\}_{k=0}^{\infty}$ and also $\left\{ \|A_k\|^{\frac{1}{k+1}} \right\}_{k=0}^{\infty}$ are sequences bounded by some $M < \infty$:

$$\|A_k\| \leq M^{k+1}, \qquad \text{for all } k \geq 0.$$

Just as before, we obtain from this by induction on n that

$$\|\Phi_n\| \leq M^n \|\Phi_0\| = M^n \sqrt{\kappa}, \qquad n \geq 0.$$

Indeed,

$$\|\Phi_n\| = \left\| \frac{1}{n} \sum_{k+\ell=n-1}^{\infty} A_k \Phi_\ell \right\| \leq \frac{1}{n} \sum_{k+\ell=n-1}^{\infty} \|A_k \Phi_\ell\| \leq \frac{1}{n} \sum_{k+\ell=n-1}^{\infty} M^{k+1} M^\ell \sqrt{\kappa} = M^n \sqrt{\kappa},$$

where this time we have used the multiplicativity of the norm on matrices, $\|AB\| \leq \|A\| \|B\|$, together with the induction assumption $\|\Phi_\ell\| \leq M^\ell \sqrt{\kappa}$ for $\ell < n$. Note that $\|\Phi_0\| = \|\kappa \times \kappa \textit{ identity}\| = \sqrt{\kappa}$.

Thus

$$\limsup_{n\to\infty} \|\Phi_n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left(M^n \sqrt{\kappa} \right)^{\frac{1}{n}} = M,$$

and so Proposition 1 shows that the matrix-valued series $\sum_{n=0}^{\infty} \Phi_n x^n$ has radius of convergence $R \geq \frac{1}{M} > 0$, and standard results on series now show that $\Phi(x) = \sum_{n=0}^{\infty} \Phi_n x^n$ is a solution to the initial value problem (1.2) with $x_0 = 0$ for $|x| < R$.

1.5. The optimal estimate on R. We end this section on the theory of power series with a sketch of the tricky induction needed to prove the optimal inequality

$$R \geq \rho$$

for the radius of convergence R of the series $\sum_{n=0}^{\infty} \Phi_n x^n$ in Theorem 11.

Start by choosing positive numbers S and T such that $\frac{1}{\rho} < S < T$, and divide the recursion relation (1.5) by T^n to obtain

$$\frac{\Phi_n}{T^n} = \frac{1}{n} \sum_{k+\ell=n-1} \left(\frac{A_k}{T^{k+1}}\right)\left(\frac{\Phi_\ell}{T^\ell}\right) = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{A_k}{T^{k+1}}\right)\left(\frac{\Phi_{n-k-1}}{T^{n-k-1}}\right).$$

Since $\limsup_{k\to\infty} \|A_k\|^{\frac{1}{k}} = \frac{1}{\rho} < S$, we have $\|A_k\| \leq S^{k+1}$ for k sufficiently large, say $k \geq m$. And of course we have the estimate established in the previous subsection in terms of M:

$$\|A_k\| \leq M^{k+1} \qquad \text{for all } k \geq 0.$$

Altogether then, for $n \geq m$ we have the estimate

$$\begin{aligned}
\left\|\frac{\Phi_n}{T^n}\right\| &\leq \frac{1}{n}\sum_{k=0}^{m-1}\left\|\frac{A_k}{T^{k+1}}\right\|\left\|\frac{\Phi_{n-k-1}}{T^{n-k-1}}\right\| + \frac{1}{n}\sum_{k=m}^{n-1}\left\|\frac{A_k}{T^{k+1}}\right\|\left\|\frac{\Phi_{n-k-1}}{T^{n-k-1}}\right\| \\
&\leq \frac{1}{n}\sum_{k=0}^{m-1}\left(\frac{M}{T}\right)^{k+1}\left\|\frac{\Phi_{n-k-1}}{T^{n-k-1}}\right\| + \frac{1}{n}\sum_{k=m}^{n-1}\left(\frac{S}{T}\right)^{k+1}\left\|\frac{\Phi_{n-k-1}}{T^{n-k-1}}\right\| \\
&\leq \left(\frac{1}{n}\sum_{k=0}^{m-1}\left(\frac{M}{T}\right)^{k+1}\right)\sup_{0\leq\ell\leq n-1}\left\|\frac{\Phi_\ell}{T^\ell}\right\| + \left(\frac{1}{n}\sum_{k=m}^{n-1}\left(\frac{S}{T}\right)^{k+1}\right)\sup_{0\leq\ell\leq n-1}\left\|\frac{\Phi_\ell}{T^\ell}\right\| \\
&= \left(\frac{1}{n}\sum_{k=0}^{m-1}\left(\frac{M}{T}\right)^{k+1} + \left(\frac{S}{T}\right)^{m+1}\frac{1}{n}\sum_{k=m}^{n-1}\left(\frac{S}{T}\right)^{k-m}\right)\sup_{0\leq\ell\leq n-1}\left\|\frac{\Phi_\ell}{T^\ell}\right\|.
\end{aligned}$$

Now with m fixed, we choose N so large that

$$\frac{1}{N}\sum_{k=0}^{m-1}\left(\frac{M}{S}\right)^{k+1} \leq 1 - \left(\frac{S}{T}\right)^{m+1}.$$

Thus using $\frac{1}{n}\sum_{k=m}^{n-1}\left(\frac{S}{T}\right)^{k-m} \leq 1$, we have for $n \geq N$ that

$$\frac{1}{n}\sum_{k=0}^{m-1}\left(\frac{M}{T}\right)^{k+1} + \left(\frac{S}{T}\right)^{m+1}\frac{1}{n}\sum_{k=m}^{n-1}\left(\frac{S}{T}\right)^{k-m} \leq \frac{1}{N}\sum_{k=0}^{m-1}\left(\frac{M}{S}\right)^{k+1} + \left(\frac{S}{T}\right)^{m+1} \leq 1,$$

and hence

$$\left\|\frac{\Phi_n}{T^n}\right\| \leq \sup_{0\leq\ell\leq n-1}\left\|\frac{\Phi_\ell}{T^\ell}\right\|, \qquad n \geq N. \tag{1.6}$$

Induction on n now shows that for $n \geq 0$ we have

$$\left\|\frac{\Phi_n}{T^n}\right\| \leq \sup_{0\leq\ell\leq N}\left\|\frac{\Phi_\ell}{T^\ell}\right\|. \tag{1.7}$$

Indeed, the inequality (1.7) is trivial for $0 \leq n \leq N$, and for $n > N$ (1.6) gives

$$\left\|\frac{\Phi_n}{T^n}\right\| \leq \sup_{0\leq\ell\leq n-1}\left\|\frac{\Phi_\ell}{T^\ell}\right\| \leq \sup_{0\leq\ell\leq N}\left\|\frac{\Phi_\ell}{T^\ell}\right\|,$$

by the induction assumption.

From the inequality (1.7), we thus obtain

$$\|\Phi_n\| \leq T^n \sup_{0\leq \ell \leq N} \left\|T^{-\ell}\Phi_\ell\right\| = CT^n,$$

where $C = \sup_{0\leq \ell \leq N} \left\|T^{-\ell}\Phi_\ell\right\|$ is independent of n, and hence

$$\limsup_{n\to\infty} \|\Phi_n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} (CT^n)^{\frac{1}{n}} = T.$$

Proposition 1 now shows that the radius of convergence R of $\sum_{n=0}^{\infty} \Phi_n x^n$ satisfies $R \geq \frac{1}{T}$. Since $T > \frac{1}{\rho}$ can be chosen arbitrarily close to $\frac{1}{\rho}$, we conclude that $R \geq \frac{1}{\frac{1}{\rho}} = \rho$.

2. Regular singular points

Every point x_0 on the real line is an ordinary point for Airy's equation

$$y'' - xy = 0, \qquad -\infty < x < \infty,$$

the prototypical example of the simplest higher order equation with variable coefficients. An almost equally simple example is the equation

$$x^2 y'' - y = 0, \qquad -\infty < x < \infty. \tag{2.1}$$

However, if we plug a power series $y = \sum_{n=0}^{\infty} c_n x^n$ centered at $x_0 = 0$ into this equation, we get

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) c_n x^n &= x^2 \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\ &= xy'' = y = \sum_{n=0}^{\infty} c_n x^n, \end{aligned}$$

and equating coefficients of like powers of x we get the recursion relation

$$\begin{aligned} 0 &= c_0, \\ 0 &= c_1, \\ n(n-1) c_n &= c_n, \qquad n \geq 2, \end{aligned}$$

which gives $c_n = 0$ for all n! Thus, apart from the trivial solution, there are *no* power series solutions centered at $x_0 = 0$ to the equation (2.1). The problem lies in the fact that 0 is *not* an ordinary point for the equation (2.1).

On the other hand, (2.1) is a Cauchy-Euler equation with indicial polynomial

$$P(\lambda) = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1$$

having roots

$$\frac{1 \pm \sqrt{1+4}}{2}.$$

We know from the previous chapter that a fundamental solution set for (2.1) with $x > 0$ is given by

$$\left\{ x^{\frac{1+\sqrt{5}}{2}}, x^{-\frac{1+\sqrt{5}}{2}} \right\}.$$

As we will see below, any equation of the form

$$x^2 a_2(x) y'' + x a_1(x) y' + a_0(x) y = 0, \tag{2.2}$$

in which the functions $a_k(x)$ are analytic at 0, and

$$a_2(0) = 1, \quad a_1(0) = 0, \quad a_0(0) = -1, \tag{2.3}$$

will have a fundamental solution set for $x > 0$ of the form

$$\left\{x^{\frac{1+\sqrt{5}}{2}} y_0(x), x^{\frac{1+\sqrt{5}}{2}} y_1(x)\right\} = \{x^{r_0} y_0(x), x^{r_1} y_1(x)\},$$

$$r_0 \equiv \frac{1+\sqrt{5}}{2}, \quad r_1 \equiv \frac{1-\sqrt{5}}{2},$$

where $y_0(x)$ and $y_1(x)$ are analytic at $x = 0$, and where moreover, the coefficients c_n^i in the power series expansions

$$y_i(x) = 1 + c_1^i x + c_2^i x^2 + ..., \qquad i = 1, 2,$$

can be computed by substitution in (2.2). Of course, if some $x_0 > 0$ is an ordinary point for (2.2), then we can find power series solutions centered at x_0, but these power series will have their intervals of convergence limited by the singular point at 0, a defect avoided by the solutions $x^{r_0} y_0(x)$ and $x^{r_1} y_1(x)$. Even more importantly, in some applications a regular singular point has special physical significance, such as in Bessel's equation for a radially vibrating circular drumskin (treated below), in which the regular singular point $x = 0$ corresponds to the center of the drumskin. In such cases, an expansion about an ordinary point, away from the point of physical significance, does not give useful information regarding the physical nature of the solutions.

Note that because of (2.3), the equation (2.2) can be considered as an *analytic perturbation* of the Cauchy-Euler equation (2.1), where the analytic function

$$a_j(x) = a_j(0) + a_j'(0)x + \frac{1}{2}a_j''(0)x^2...$$

replaces the constant function $a_j(0)$ for $j = 2, 1, 0$. Then we can also consider the fundamental solution

$$\begin{aligned} x^{r_i} y_i(x) &= x^{r_i} + c_1^i x^{r_i+1} + c_2^i x^{r_i+2} + ... \\ &= x^{r_i}\left(1 + c_1^i x + c_2^i x^2 + ...\right) \end{aligned}$$

to (2.2), as an *analytic perturbation* of the fundamental solution x^{r_i} to the Cauchy-Euler equation (2.1).

We restrict our attention here to *second* order homogeneous linear equations,

$$A(x) y'' + B(x) y' + C(x) y = 0, \tag{2.4}$$

with variable coefficients. The reader should have no trouble however, in extending the methods below to higher order equations. Recall that a point x_0 is said to be *singular* for the equation (2.4) if it is *not* ordinary, i.e. if it is *not* the case that $A(x)$, $B(x)$ and $C(x)$ are analytic at x_0 with $A(x_0) \neq 0$.

The nicest case of a singular point x_0 is when $A(x)$, $B(x)$ and $C(x)$ are analytic at x_0, and when in normal form,

$$y'' + \frac{B(x)}{A(x)} y' + \frac{C(x)}{A(x)} y = 0, \tag{2.5}$$

we can factor $(x - x_0)^{-1}$ out of $\frac{B(x)}{A(x)}$ and $(x - x_0)^{-2}$ out of $\frac{C(x)}{A(x)}$, i.e.

$$\frac{B(x)}{A(x)} = (x - x_0)^{-1} p(x) \text{ and } \frac{C(x)}{A(x)} = (x - x_0)^{-2} q(x), \tag{2.6}$$

where $p(x)$ and $q(x)$ are analytic at x_0.

Definition 5. *We say that x_0 is a* regular singular *point for the equation (2.4) if x_0 is not an ordinary point, and if (2.6) holds. Thus at a regular singular point x_0, the normal form (2.5) can be multiplied by $(x-x_0)^2$ and put in the* Cauchy-Euler form*,*

$$(x-x_0)^2 y'' + (x-x_0)\, p(x)\, y' + q(x)\, y = 0. \tag{2.7}$$

It turns out that a fundamental solution set for equation (2.7) is typically an analytic perturbation of a fundamental solution set for the associated Cauchy-Euler equation

$$(x-x_0)^2 y'' + (x-x_0)\, p(x_0)\, y' + q(x_0)\, y = 0.$$

There is however, the possibility of a *wrinkle* when the roots of the indicial polynomial

$$P(\lambda) = \lambda(\lambda-1) + p(x_0)\lambda + q(x_0) = 0$$

differ by exactly an integer. To see what is going on here, we investigate the case $x_0 = 0$ when the roots r_1 and r_2 are real.

Consider the equation

$$x^2 y'' + x p(x)\, y' + q(x)\, y = 0, \tag{2.8}$$

where

$$p(x) = \sum_{k=0}^{\infty} p_k x^k \text{ and } q(x) = \sum_{k=0}^{\infty} q_k x^k$$

are analytic at 0, and the associated indicial polynomial

$$\begin{aligned} P(\lambda) &= \lambda(\lambda-1) + p_0\lambda + q_0 = (\lambda - r_1)(\lambda - r_2), \\ r_1 &\geq r_2, \end{aligned}$$

has two real roots with $r_1 \geq r_2$. Motivated by the fact that the Cauchy-Euler equation

$$x^2 y'' + x p_0 y' + q_0 y = 0, \tag{2.9}$$

has a fundamental solution set

$$\begin{cases} \{x^{r_1}, x^{r_2}\} & \text{when} \quad r_1 > r_2 \\ \{x^{r_1}, x^{r_1}\ln x\} & \text{when} \quad r_1 = r_2 \end{cases},$$

we take $y(x)$ to be a series of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r},$$

which we think of as an analytic perturbation of x^r. Using the products

$$\begin{aligned} q(x)\, y(x) &= \left(\sum_{k=0}^{\infty} q_k x^k\right)\left(\sum_{\ell=0}^{\infty} c_\ell x^{\ell+r}\right) = \sum_{n=0}^{\infty}\left(\sum_{k+\ell=n} q_k c_\ell\right) x^{n+r}, \\ x p(x)\, y'(x) &= x\left(\sum_{k=0}^{\infty} p_k x^k\right)\left(\sum_{\ell=0}^{\infty} (\ell+r)\, c_\ell x^{\ell+r-1}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k+\ell=n} p_k (\ell+r)\, c_\ell\right) x^{n+r}, \end{aligned}$$

we plug $y(x)$ into equation (2.8) to obtain

$$\begin{aligned}
0 &= x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \\
&\quad + \sum_{n=0}^{\infty} \left(\sum_{k+\ell=n} p_k (\ell+r) c_\ell \right) x^{n+r} + \sum_{n=0}^{\infty} \left(\sum_{k+\ell=n} q_k c_\ell \right) x^{n+r} \\
&= \sum_{n=0}^{\infty} \left\{ (n+r)(n+r-1) c_n + \sum_{k+\ell=n} p_k (\ell+r) c_\ell + \sum_{k+\ell=n} q_k c_\ell \right\} x^{n+r}.
\end{aligned}$$

Now in the expression in braces above, the coefficient of highest index, namely c_n, occurs in three places, and collecting these three terms, we see that c_n is mulitplied by

$$(n+r)(n+r-1) + p_0 (n+r) + q_0 = P(n+r),$$

the indicial polynomial $P(\lambda)$ evalutated at $\lambda = n+r$. If we equate coefficients of like powers of x we thus get the recursion relation

$$\begin{aligned}
0 &= P(r) c_0, \\
0 &= P(n+r) c_n + \sum_{\substack{k+\ell=n \\ \ell<n}} p_k (\ell+r) c_\ell + \sum_{\substack{k+\ell=n \\ \ell<n}} q_k c_\ell, \qquad n \geq 1,
\end{aligned}$$

which can be solved for c_n when $P(n+r) \neq 0$:

$$\begin{aligned}
0 &= P(r) c_0, \\
c_n &= -\frac{1}{P(n+r)} \left(\sum_{\substack{k+\ell=n \\ \ell<n}} p_k (\ell+r) c_\ell + \sum_{\substack{k+\ell=n \\ \ell<n}} q_k c_\ell \right), \qquad n \geq 1.
\end{aligned}$$

At this point we should take note of the important role played by the indicial polynomial $P(\lambda)$ associated with the Cauchy-Euler equation (2.9).

If we take $r = r_1$, then $P(r_1) = 0$ and the recursion relation leaves c_0 unrestricted. Moreover, since r_1 is the *largest* real root, $P(n+r) \neq 0$ for all $n \geq 1$, and the recursion relation inductively determines all of the coefficients c_n uniquely in terms of c_0. It can then be shown that the power series portion of

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} c_n x^n$$

has optimal positive radius of convergence, and that the series $y_1(x)$ is in fact a solution to the equation (2.8).

Now we take $r = r_2$ in $y(x) = x^r \sum_{n=0}^{\infty} c_n x^n$, so that again $P(r_2) = 0$, and the recursion relation leaves c_0 unrestricted. But now the wrinkle mentioned above makes its appearance. If the difference of the roots $r_1 - r_2$ is a positive *integer* N, then we cannot in general solve for the N^{th} coefficient c_N in the recursion relation

$$c_N = -\frac{1}{P(N+r_2)} \left(\sum_{\substack{k+\ell=N \\ \ell<N}} p_k (\ell+r_2) c_\ell + \sum_{\substack{k+\ell=N \\ \ell<N}} q_k c_\ell \right).$$

In fact if the complicated expression in curly brackets

$$E_N(r_2) \equiv \sum_{\substack{k+\ell=N \\ \ell<N}} p_k(\ell+r_2)c_\ell + \sum_{\substack{k+\ell=N \\ \ell<N}} q_k c_\ell \tag{2.10}$$

is nonzero, then the method stalls and there is no second independent solution of the form $y_2(x) = x^{r_2}\sum_{n=0}^{\infty} c_n x^n$. On the other hand, if we happen to be lucky enough that $E_N(r_2) = 0$, then the recursion relation leaves the coefficient c_N unrestricted, and there is indeed a second linearly independent series solution of the form

$$y_2(x) = x^{r_2}\sum_{n=0}^{\infty} c_n x^n.$$

It turns out that in the (as yet) unresolved cases,

- $r_1 - r_2 = 0$,
- $r_1 - r_2 = N \in \mathbb{N}$ with $E_N(r_2) \neq 0$,

there is a second linearly independent solution having the form

$$y_2(x) = a(\ln x)\, y_1(x) + x^{r_1}\sum_{n=0}^{\infty} d_n x^n.$$

This is certainly not surprising when $r_1 = r_2$, given that the Cauchy-Euler equation (2.9) has fundamental solution set $\{x^{r_1}, x^{r_1}\ln x\}$ in this case. The wrinkle is that for general analytic coefficients $p(x)$ and $q(x)$, a log factor can also arise in the second solution for (2.8) when the roots of the indicial polynomial differ by an integer.

Here is our theorem on fundamental solution sets centered at a regular singular point. For convenience we state the theorem with $x_0 = 0$.

THEOREM 12. *Suppose that $x = 0$ is a regular singular point for the equation*

$$x^2 y'' + xp(x)\, y' + q(x)\, y = 0.$$

Let r_1 and r_2 be the roots (either both real, possibly equal, or a complex conjugate pair) of the indicial polynomial

$$P(\lambda) = \lambda(\lambda-1) + p(0)\lambda + q(0).$$

(1) *If r_1 and r_2 are real and do not differ by an integer, then there is a fundamental solution set of the form*

$$\left\{ y_1 = x^{r_1}\sum_{n=0}^{\infty} c_n x^n, \quad y_2 = x^{r_2}\sum_{n=0}^{\infty} d_n x^n \right\}, \qquad x > 0,$$

where the power series have optimal positive radius of convergence, and the coefficients c_n and d_n can be evaluated by substituting the series in the equation, and deriving a recurrence relation for the coefficients.

(2) *If $r_1 = r_2$ is real, then there is a fundamental solution set of the form*

$$\left\{ y_1 = x^{r_1}\sum_{n=0}^{\infty} c_n x^n, \quad y_2 = (\ln x)\, y_1(x) + x^{r_1}\sum_{n=1}^{\infty} d_n x^n \right\}, \qquad x > 0,$$

where the power series have optimal positive radius of convergence, and the coefficients c_n and d_n can be evaluated by substituting the series in the equation, and deriving a recurrence relation for the coefficients (note the second series starts at $n = 1$).

(3) *If r_1 and r_2 are real and $r_1 - r_2$ is a positive integer, then there is a fundamental solution set of the form*

$$\left\{ y_1 = x^{r_1} \sum_{n=0}^{\infty} c_n x^n, \quad y_2 = a \left(\ln x\right) y_1 \left(x\right) + x^{r_2} \sum_{n=0}^{\infty} d_n x^n \right\}, \qquad x > 0,$$

where the power series have optimal positive radius of convergence, and the coefficients a, c_n and d_n can be evaluated by substituting the series in the equation, and deriving a recurrence relation for the coefficients.

(4) *If r_1 and r_2 are a complex conjugate pair $\alpha \pm i\beta$, then there is a fundamental solution set of the form*

$$\left\{ x^{\alpha} \cos\left(\beta \ln x\right) \sum_{n=0}^{\infty} c_n x^n, \quad x^{\alpha} \sin\left(\beta \ln x\right) \sum_{n=0}^{\infty} d_n x^n \right\}, \qquad x > 0,$$

where the power series have optimal positive radius of convergence. The coefficients c_n and d_n can be evaluated by substituting the series $\sum_{n=0}^{\infty} c_n x^{n+r}$ in the equation, deriving a recurrence relation for the complex-valued coefficients when $r = \alpha \pm i\beta$, and then taking real and imaginary parts of the resulting series.

We will not prove this theorem, but instead give a couple of examples to illustrate the application of parts (1) and (2). Further applications of parts (2) and (3) of the theorem are given in the next section on Bessel's equation.

Example 31. *The equation*

$$2x^2 y'' - xy' + \left(1 + x\right) y = 0, \qquad x > 0,$$

has a regular singular point at $x = 0$, and the associated indicial polynomial is

$$\begin{aligned} P\left(\lambda\right) &= 2\lambda\left(\lambda - 1\right) - \lambda + 1 = 2\lambda^2 - 3\lambda + 1 \\ &= \left(2\lambda - 1\right)\left(\lambda - 1\right), \end{aligned}$$

and has real roots $r_1 = 1$ and $r_2 = \frac{1}{2}$. So part (1) of Theorem 12 above guarantees a fundamental solution set $\{y_1, y_2\}$ of the form

$$y_1 = x \sum_{n=0}^{\infty} c_n x^n, \qquad y_2 = x^{\frac{1}{2}} \sum_{n=0}^{\infty} d_n x^n, \qquad x > 0.$$

If we substitute $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ *into the equation, shift indices in the final sum, and collect terms, we get*

$$\begin{aligned}
0 &= 2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \\
&\quad - x \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+1} \\
&= \sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n+r} \\
&\quad - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} + \sum_{n=1}^{\infty} c_{n=1} x^{n+r} \\
&= P(r) c_0 + \sum_{n=1}^{\infty} \{ P(r+n) c_n + c_{n-1} \} x^{n+r},
\end{aligned}$$

and hence, equating coefficients, we obtain the recurrence relation

$$c_n = -\frac{1}{P(r+n)} c_{n-1}, \qquad n \geq 1.$$

Solving the recurrence we get

$$\begin{aligned}
c_n &= \frac{(-1)^n}{P(r+n) P(r+n-1) \ldots P(r+1)} c_0 \\
&= \frac{(-1)^n}{(2(r+n)-1)(r+n-1) \ldots (2(r+1)-1)(r+1-1)} c_0 \\
&= \frac{(-1)^n}{(2r+2n-1) \ldots (2r+1) \cdot (r+n-1) \ldots (r)} c_0, \qquad n \geq 1.
\end{aligned}$$

Now when $r = 1$ *we get*

$$\begin{aligned}
c_n &= \frac{(-1)^n}{(2n+1)(2n-1) \ldots (3) \cdot (n) \ldots (1)} c_0 \\
&= \frac{(-1)^n}{(2n+1)!!\; n!} c_0, \qquad n \geq 1,
\end{aligned}$$

and when $r = \frac{1}{2}$ *a similar calculation gives*

$$c_n = \frac{(-1)^n}{(2n-1)!!\; n!} c_0, \qquad n \geq 1.$$

Thus we have

$$\begin{aligned}
y_1 &= x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!!\; n!} x^n, \\
y_2 &= x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)!!\; n!} x^n, \qquad x > 0.
\end{aligned}$$

EXAMPLE 32. *The equation*

$$x^2 y'' - x y' + (1-x) y = 0, \qquad x > 0,$$

has a regular singular point at $x = 0$, and the associated indicial polynomial is

$$P(\lambda) = \lambda(\lambda - 1) - \lambda + 1 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2,$$

and has repeated real roots $r_1 = r_2 = 1$. So part (2) of Theorem 12 above guarantees a fundamental solution set $\{y_1, y_2\}$ of the form

$$y_1 = x\sum_{n=0}^{\infty} c_n x^n, \qquad y_2 = (\ln x)\, y_1(x) + x\sum_{n=0}^{\infty} d_n x^n, \qquad x > 0.$$

Proceeding as in the example above we obtain that for a series $\sum_{n=0}^{\infty} c_n x^{n+r}$ to be a solution, we must have the recurrence relation

$$c_n = \frac{1}{(n + r - 1)^2} c_{n-1}, \qquad n \geq 1,$$

and so

$$c_n = \frac{1}{(n!)^2} c_0, \qquad n \geq 1.$$

Thus a first solution is

$$y_1(x) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^{n+1}, \tag{2.11}$$

and a second independent solution has the form

$$y_2(x) = (\ln x)\, y_1(x) + \sum_{n=1}^{\infty} d_n x^{n+1}.$$

Substituting y_2 into the equation gives

$$\begin{aligned} 0 \;=\;& x^2 \left\{ y_1''(x) \ln x - x^{-2} y_1(x) + 2x^{-1} y_1'(x) + \sum_{n=1}^{\infty} (n+1)\, n d_n x^{n-1} \right\} \\ & - x \left\{ y_1'(x) \ln x + x^{-1} y_1(x) + \sum_{n=1}^{\infty} (n+1)\, d_n x^n \right\} \\ & + (1 - x) \left\{ y_1(x) \ln x + \sum_{n=1}^{\infty} d_n x^{n+1} \right\}, \end{aligned}$$

equivalently

$$\begin{aligned} 0 \;=\;& \left\{ \mathbf{x^2 y_1''(x)} - \mathbf{x y_1'(x)} + \mathbf{(1 - x)\, y_1(x)} \right\} \ln x - 2y_1(x) + 2x y_1'(x) \\ & + \sum_{n=1}^{\infty} n(n+1)\, d_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)\, d_n x^{n+1} + \sum_{n=1}^{\infty} d_n x^{n+1} - \sum_{n=1}^{\infty} d_n x^{n+2}. \end{aligned}$$

Now the term in boldface type in the braces vanishes because y_1 is a solution, and hence shifting indices, and then substituting the series (2.11) for y_1, we get

$$\begin{aligned} 0 &= 2xy_1'(x) - 2y_1(x) + d_1x^2 + \sum_{n=2}^{\infty}\left(n^2d_n - d_{n-1}\right)x^{n+1} \\ &= 2\sum_{n=0}^{\infty}\frac{(n+1)}{(n!)^2}x^{n+1} - 2\sum_{n=0}^{\infty}\frac{1}{(n!)^2}x^{n+1} + d_1x^2 + \sum_{n=2}^{\infty}\left(n^2d_n - d_{n-1}\right)x^{n+1} \\ &= (2+d_1)x^2 + \sum_{n=2}^{\infty}\left(\frac{2n}{(n!)^2} + n^2d_n - d_{n-1}\right)x^{n+1}, \end{aligned}$$

which gives the recurrence relation

$$\begin{aligned} d_1 &= -2, \\ d_n &= \frac{1}{n^2}\left(d_{n-1} - \frac{2n}{(n!)^2}\right), \qquad n \geq 2. \end{aligned}$$

Thus we have

$$d_2 = \frac{1}{4}(d_1 - 1) = -\frac{3}{4}, \quad d_3 = \frac{1}{9}\left(-\frac{3}{4} - \frac{6}{36}\right) = -\frac{11}{108},$$

and so a second linearly independent solution is given by

$$y_2(x) = y_1(x)\ln x - 2x^2 - \frac{3}{4}x^3 - \frac{11}{108}x^4 + ...,$$

where we will not solve for the remaining coefficients d_n explicitly.

3. Bessel's equation

Bessel's equation of order ν, where ν is a real constant, is

$$L[y] \equiv x^2y'' + xy' + \left(x^2 - \nu^2\right)y = 0, \qquad x > 0, \tag{3.1}$$

and has a regular singular point at $x = 0$, and indicial polynomial

$$P(\lambda) = \lambda(\lambda - 1) + \lambda - \nu^2 = \lambda^2 - \nu^2 = (\lambda - \nu)(\lambda + \nu),$$

with roots $\pm\nu$. For convenience, we are here considering series expansions only for $x > 0$. We will show that the cases $\nu = 0$, $\nu = \frac{1}{2}$, and $\nu = 1$ of Bessel's equation (3.1) illustrate respectively application of parts (2), (3) with $a = 0$, and (3) with $a \neq 0$ of Theorem 12.

Before proceeding with these calculations, we informally discuss the qualititative behavior of solutions to (3.1) for x large. First we rewrite the equation (3.1) in normal form,

$$y'' + \left(\frac{1}{x}\right)y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0,$$

and note that for x large, both $\frac{1}{x} \approx 0$ and $\frac{\nu^2}{x^2} \approx 0$. Thus we might expect that for x large, solutions to Bessel's equation (3.1) behave qualititatively like solutions to the equation

$$\begin{aligned} y'' + (0)y' + (1 - 0)y &= 0; \\ y'' + y &= 0. \end{aligned}$$

The general solution to the latter equation is

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x = A \cos (x - \theta) ; \\ A &= \sqrt{c_1^2 + c_2^2}, \quad \theta = \tan^{-1} \frac{c_2}{c_1}, \end{aligned}$$

which oscillates back and forth between A and $-A$ with period 2π. We will see below that this is approximately true of solutions to Bessel's equation (3.1) when x is large.

3.1. The order $\nu = 0$ case. In the case $\nu = 0$ the indicial polynomial has repeated root 0. Thus part (2) of Theorem 12 gives a fundamental solution set of the form

$$(3.2) \qquad \left\{ y_1 = \sum_{n=0}^{\infty} c_n x^n, \quad y_2 = (\ln x) \, y_1 (x) + \sum_{n=1}^{\infty} d_n x^n \right\}, \qquad x > 0.$$

Since

$$\begin{aligned} (3.3) \quad 0 &= L \left[\sum_{n=0}^{\infty} c_n x^{n+r} \right] = \sum_{n=0}^{\infty} (n+r)(n+r-1) \, c_n x^{n+r} \\ &\quad + \sum_{n=0}^{\infty} (n+r) \, c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+2+r} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \left(r^2 - \nu^2 \right) c_0 x^r + \left((r+1)^2 - \nu^2 \right) c_1 x^{1+r} \\ &\quad + \sum_{n=2}^{\infty} \left\{ \left[(n+r)^2 - \nu^2 \right] c_n + c_{n-2} \right\} x^{n+r}, \end{aligned}$$

we obtain the following recursion relation when $r = \nu = 0$:

$$\begin{aligned} c_1 &= 0, \\ c_n &= -\frac{1}{n^2} c_{n-2}, \qquad n \geq 2. \end{aligned}$$

Thus we have both

$$\begin{aligned} c_{2k} &= -\frac{1}{(2k)^2} c_{2k-2} = \left(-\frac{1}{(2k)^2} \right) \left(-\frac{1}{(2k-2)^2} \right) c_{2k-4} \\ &= \ldots = (-1)^k \frac{1}{(2k)^2 (2k-2)^2 \ldots 2^2} c_0 \\ &= \frac{(-1)^k}{2^{2k} (k!)^2} c_0, \qquad k \geq 0, \end{aligned}$$

and

$$c_{2k+1} = 0, \qquad k \geq 0.$$

It is customary to denote the power series solution $y_1(x)$ constructed above by

$$J_0(x) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} (k!)^2} x^{2k} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{64 \cdot 36} + \ldots, \qquad x > 0,$$

and to refer to $J_0(x)$ as the Bessel function of the first kind of order 0.

The second solution in the fundamental solution set (3.2) has the form

$$y_2 = (\ln x)\, J_0\,(x) + \sum_{n=1}^{\infty} d_n x^n.$$

We compute that

$$\begin{aligned}
\left[(\ln x)\, J_0\,(x)\right]' &= (\ln x)\, J_0'\,(x) + \frac{1}{x} J_0\,(x)\,; \\
\left[(\ln x)\, J_0\,(x)\right]'' &= (\ln x)\, J_0''\,(x) + \frac{2}{x} J_0'\,(x) - \frac{1}{x^2} J_0\,(x)\,,
\end{aligned}$$

and so

$$\begin{aligned}
L\left[(\ln x)\, J_0\,(x)\right] &= x^2 \left\{ (\ln x)\, J_0''\,(x) + \frac{2}{x} J_0'\,(x) - \frac{1}{x^2} J_0\,(x) \right\} \\
&\quad + x \left\{ (\ln x)\, J_0'\,(x) + \frac{1}{x} J_0\,(x) \right\} + \left(x^2 - \nu^2\right)(\ln x)\, J_0\,(x) \\
&= (\ln x) \left\{ \mathbf{x}^2 \mathbf{J}_0''\,(\mathbf{x}) + \mathbf{x}\mathbf{J}_0'\,(\mathbf{x}) + \left(\mathbf{x}^2 - \boldsymbol{\nu}^2\right) \mathbf{J}_0\,(\mathbf{x}) \right\} + 2x J_0'\,(x) \\
&= (\ln x)\, L\left[J_0\,(x)\right] + 2x J_0'\,(x) = 2x J_0'\,(x)\,, \qquad \nu = 0.
\end{aligned}$$

Plugging y_2 into Bessel's equation (3.1), and using the calculation in (3.3) with $\nu = r = 0$, we thus obtain

$$\begin{aligned}
0 &= L\left[(\ln x)\, J_0\,(x) + \sum_{n=1}^{\infty} d_n x^n\right] = L\left[(\ln x)\, J_0\,(x)\right] + L\left[\sum_{n=1}^{\infty} d_n x^n\right] \\
&= 2x J_0'\,(x) + d_1 x + \sum_{n=2}^{\infty} \left\{ n^2 d_n + d_{n-2} \right\} x^n.
\end{aligned}$$

Now we substitute the series for

$$J_0'\,(x) = \sum_{k=1}^{\infty} \frac{(-1)^k\,(2k)}{2^{2k}\,(k!)^2} x^{2k-1}$$

into this equation to get

$$-2 \sum_{k=1}^{\infty} \frac{(-1)^k\,(2k)}{2^{2k}\,(k!)^2} x^{2k} = -2x J_0'\,(x) = d_1 x + \sum_{n=2}^{\infty} \left\{ n^2 d_n + d_{n-2} \right\} x^n.$$

Since only *even* powers of x appear on the left side of this equation, it follows that all of the coefficients of *odd* powers of x on the right side must vanish, i.e.

$$\begin{aligned}
0 &= d_1, \\
0 &= (2k+1)^2\, d_{2k+1} + d_{2k-1}, \qquad k \geq 2.
\end{aligned}$$

By induction, we obtain from this that all the *odd*-indexed coefficients d_{2k+1} vanish:

$$d_{2k+1} = 0, \qquad k \geq 0.$$

As for the *even*-indexed coefficients d_{2k}, we have $d_0 = 0$ and

$$\begin{aligned}
(2k)^2\, d_{2k} + d_{2k-2} &= -2 \frac{(-1)^k\,(2k)}{2^{2k}\,(k!)^2}, \qquad k \geq 1; \\
d_{2k} &= -\frac{(-1)^k}{k 2^{2k}\,(k!)^2} - \frac{1}{(2k)^2} d_{2k-2}, \qquad k \geq 1.
\end{aligned}$$

The first few even-indexed coefficients are thus given by

$$\begin{aligned}
d_2 &= -\frac{(-1)}{2^2(1!)^2} - \frac{1}{(2)^2}d_0 = \frac{1}{2^2},\\
d_4 &= -\frac{(-1)^2}{2\cdot 2^4(2!)^2} - \frac{1}{4^2}d_2\\
&= -\frac{1}{2}\frac{1}{4^2(2!)^2} - \frac{1}{4^2}\frac{1}{2^2(1!)^2} = -\frac{1}{2^24^2}\left(1+\frac{1}{2}\right),\\
d_6 &= -\frac{(-1)^k}{k2^{2k}(k!)^2} - \frac{1}{(2k)^2}d_{2k-2} = -\frac{(-1)^3}{3\cdot 2^6\cdot 6^2} - \frac{1}{6^2}d_4\\
&= \frac{1}{3\cdot 2^6\cdot 6^2} + \frac{1}{6^2}\frac{1}{2^24^2}\left(1+\frac{1}{2}\right)\\
&= \frac{1}{2^24^26^2}\left(1+\frac{1}{2}+\frac{1}{3}\right),
\end{aligned}$$

and we are led to guess the following formula, which is easily proved by induction on k:

$$d_{2k} = \frac{(-1)^{k+1}}{2^{2k}(k!)^2}H_k, \qquad k \geq 1,$$

where

$$H_k \equiv 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k} = \sum_{m=1}^{k}\frac{1}{m}.$$

Thus a second independent solution is given by

$$y_2(x) = (\ln x)J_0(x) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_k}{2^{2k}(k!)^2}x^{2k}, \qquad x > 0.$$

It is customary to define the Bessel function $Y_0(x)$ of the second kind of order zero by

$$Y_0(x) \equiv \frac{2}{\pi}\left\{y_2(x) + (\gamma - \ln 2)J_0(x)\right\},$$

where the Euler-Máscheroni constant γ is given by

$$\gamma \equiv \lim_{k\to\infty}(H_k - \ln k) \approx 0.5772.$$

Altogether we have

$$Y_0(x) = \frac{2}{\pi}\left\{\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_k}{2^{2k}(k!)^2}x^{2k}\right\}, \qquad x > 0.$$

The general solution of Bessel's equation when $\nu = 0$ is thus

$$y(x) = c_1J_0(x) + c_2Y_0(x), \qquad x > 0.$$

3.1.1. *Qualitative behaviour of J_0 and Y_0.* For $x > 0$ and close to 0, $J_0(x)$ behaves like 1 and $Y_0(x)$ behaves like $\frac{2}{\pi}\ln x$, in the sense that

$$\lim_{x\to 0^+} J_0(x) = 1 \text{ and } \lim_{x\to 0^+} \frac{Y_0(x)}{\frac{2}{\pi}\ln x} = 1.$$

For x large, the discussion at the beginning of the section indicated that both $J_0(x)$ and $Y_0(x)$ should oscillate regularly as $x \to \infty$. This is in fact true, but there is also an inverse square root decay as $x \to \infty$, and a more delicate analysis gives the following asymptotics at infinity:

$$\lim_{x\to\infty} \frac{J_0(x)}{\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)} = 1 \text{ and } \lim_{x\to\infty} \frac{Y_0(x)}{\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right)} = 1.$$

3.2. The order $\nu = \frac{1}{2}$ case. When $\nu = \frac{1}{2}$ the Bessel equation is

$$L[y] = x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0, \tag{3.4}$$

and the roots of the indicial polynomial are $\pm\frac{1}{2}$. Thus the roots differ by exactly 1, and part (3) of Theorem 12 applies to show there is a fundamental solution set of the form

$$\left\{ y_1 = x^{\frac{1}{2}} \sum_{n=0}^{\infty} c_n x^n, \quad y_2 = a(\ln x)\, y_1(x) + x^{-\frac{1}{2}} \sum_{n=0}^{\infty} d_n x^n \right\}, \qquad x > 0.$$

It turns out that in the situation at hand, the constant a above vanishes, and the second solution has the simpler form $y_2 = x^{-\frac{1}{2}} \sum_{n=0}^{\infty} d_n x^n$. Here is a brief sketch of the details.

From (3.3) with $\nu = \frac{1}{2}$ we obtain

$$\begin{aligned} L\left[\sum_{n=0}^{\infty} c_n x^{n+r}\right] &= \left(r^2 - \frac{1}{4}\right) c_0 x^r + \left((r+1)^2 - \frac{1}{4}\right) c_1 x^{1+r} \\ &\quad + \sum_{n=2}^{\infty} \left\{ \left[(n+r)^2 - \frac{1}{4}\right] c_n + c_{n-2} \right\} x^{n+r}. \end{aligned} \tag{3.5}$$

With $r = \frac{1}{2}$ we then obtain the recurrence relation

$$\begin{aligned} c_1 &= 0, \\ c_n &= -\frac{1}{n(n+1)} c_{n-2}, \qquad n \geq 2, \end{aligned}$$

and with a little calculation we get

$$\begin{aligned} c_{2k} &= \frac{(-1)^k}{(2k+1)!} c_0, \qquad k \geq 0, \\ c_{2k+1} &= 0, \qquad k \geq 0. \end{aligned}$$

Taking $c_0 = 1$ we thus get the solution

$$y_1(x) = x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}.$$

But if we factor out an additional power of x from the infinite sum, we recognize the resulting series as the Taylor series for $\sin x$ at the origin:

$$y_1(x) = x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x^{-\frac{1}{2}} \sin x.$$

It is customary to define the Bessel function $J_{\frac{1}{2}}(x)$ of the first kind of order $\frac{1}{2}$ by

$$J_{\frac{1}{2}}(x) \equiv \sqrt{\frac{2}{\pi x}} \sin x, \qquad x > 0.$$

Now we turn to the case when $r = -\frac{1}{2}$ is the smaller of the two roots. The first thing we observe regarding the calculation (3.5), is that both $\left(r^2 - \frac{1}{4}\right)$ and $\left((r+1)^2 - \frac{1}{4}\right)$ vanish when $r = -\frac{1}{2}$, so that both of the coefficients c_0 and c_1 are left unrestricted, and with a little bit of work, the recursion relation leads to the solution

$$\begin{aligned} y_2(x) &= x^{-\frac{1}{2}} \left\{ c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right\} \\ &= c_0 x^{-\frac{1}{2}} \cos x + c_1 x^{-\frac{1}{2}} \sin x. \end{aligned}$$

Note that the expression in (2.10) is

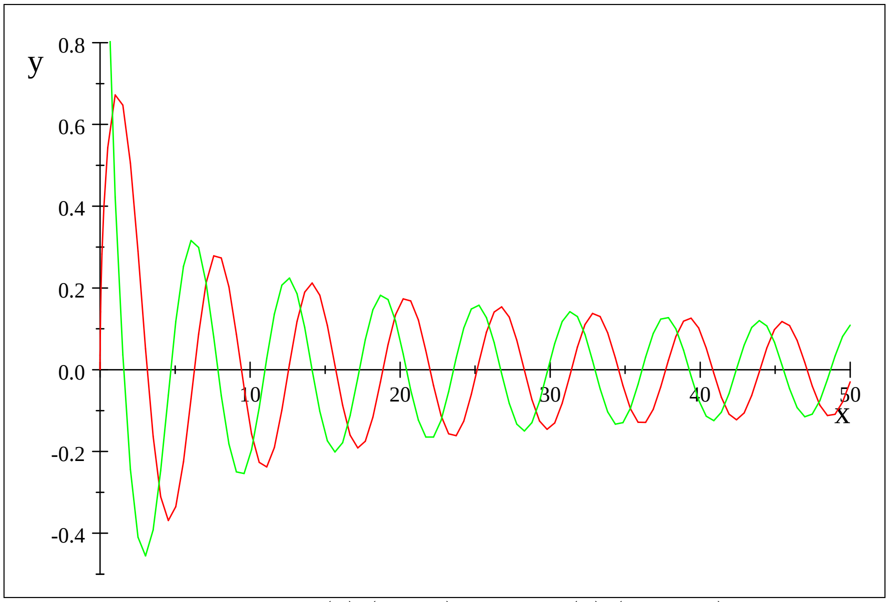
$$\begin{aligned} E_1\left(-\frac{1}{2}\right) &= \sum_{\substack{k+\ell=1 \\ \ell<1}} p_k \left(\ell - \frac{1}{2}\right) c_\ell + \sum_{\substack{k+\ell=1 \\ \ell<1}} q_k c_\ell \\ &= \left\{ p_1 \left(-\frac{1}{2}\right) + q_1 \right\} c_0 = \left\{ 0 \left(-\frac{1}{2}\right) + 0 \right\} c_0 = 0, \end{aligned}$$

which is consistent with the absence of a log factor in the second independent solution. If we take $c_1 = 0$ and $c_0 = \sqrt{\frac{2}{\pi}}$, we obtain the following function for the second solution to (3.4),

$$J_{-\frac{1}{2}}(x) \equiv \sqrt{\frac{2}{\pi x}} \cos x, \qquad x > 0,$$

referred to as the Bessel function of the first kind of order $-\frac{1}{2}$. The general solution of Bessel's equation when $\nu = \frac{1}{2}$ is thus

$$y(x) = c_1 J_{\frac{1}{2}}(x) + c_2 J_{-\frac{1}{2}}(x), \qquad x > 0.$$



Graphs of $J_{\frac{1}{2}}(x)$ (in red) and $J_{-\frac{1}{2}}(x)$ (in green)

3.3. The order $\nu = 1$ case. When $\nu = 1$ the Bessel equation is

$$L[y] = x^2 y'' + xy' + \left(x^2 - 1\right) y = 0,$$

and the roots of the indicial polynomial are ± 1. Thus the roots differ by exactly 2, and part (3) of Theorem 12 applies to show there is a fundamental solution set of the form

$$\left\{ y_1 = x \sum_{n=0}^{\infty} c_n x^n, \quad y_2 = a (\ln x) y_1(x) + x^{-1} \sum_{n=0}^{\infty} d_n x^n \right\}, \qquad x > 0. \tag{3.6}$$

It turns out that this time, the constant a above doesn *not* vanish, and the second solution has a log term. Here is a very brief sketch of the details.

From (3.3) with $\nu = 1$ we obtain

$$L\left[\sum_{n=0}^{\infty} c_n x^{n+r}\right] = \left(r^2 - 1\right) c_0 x^r + \left((r+1)^2 - 1\right) c_1 x^{1+r} + \sum_{n=2}^{\infty} \left\{ \left[(n+r)^2 - 1\right] c_n + c_{n-2} \right\} x^{n+r}.$$

With $r = 1$ we then obtain the recurrence relation

$$\begin{aligned} c_1 &= 0, \\ c_n &= -\frac{1}{n(n+2)} c_{n-2}, \qquad n \geq 2, \end{aligned}$$

and with a little calculation we get

$$\begin{aligned} c_{2k} &= \frac{(-1)^k}{2^{2k}(k+1)!k!} c_0, \qquad k \geq 0, \\ c_{2k+1} &= 0, \qquad k \geq 0. \end{aligned}$$

Taking $c_0 = \frac{1}{2}$ we get the solution

$$J_1(x) = \frac{x}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k+1)!k!} x^{2k},$$

referred to as the Bessel function of the first kind of order 1.

In order to compute the coefficients d_n in the second independent solution y_2 in (3.6), we can proceed as in the calculation of Y_0 above. The result is that after much computation, and with the choice $d_1 = \frac{1}{4}$, we get

$$y_2(x) = -J_1(x) \ln x + \frac{1}{x} \left\{ 1 + \frac{1}{4} x^2 - \sum_{k=2}^{\infty} \frac{(-1)^k (H_k + H_{k-1})}{2^{2k} k! (k-1)!} x^{2k} \right\}, \qquad x > 0.$$

It is customary to define

$$Y_1(x) = \frac{2}{\pi} \{ -y_2(x) + (\gamma - \ln 2) J_1(x) \}, \qquad x > 0,$$

referred to as the Bessel function of the second kind of order 1. The general solution of Bessel's equation when $\nu = 1$ is thus

$$y(x) = c_1 J_1(x) + c_2 Y_1(x), \qquad x > 0.$$

3.4. The case 2ν not an integer. It is convenient to recall the Gamma function $\Gamma(s)$ at this point. It is defined initially for $s > 0$ by the convergent improper integral

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt, \qquad s > 0.$$

The main interest in the Gamma function is that it satisfies the identity

$$\Gamma(n+1) = n!, \qquad n = 0, 1, 2, 3, ...$$

and arises in many series expansions, such as in the binomial theorem

$$(1-x)^s = \sum_{n=0}^{\infty} \binom{s}{n} x^n = \sum_{n=0}^{\infty} \frac{\Gamma(s+1)}{\Gamma(s-n+1)\, n!} x^n, \qquad |x| < 1,$$

and in the expansions for Bessel functions below.

In fact, $\Gamma(s)$ satisfies the important functional equation

$$\begin{aligned} \Gamma(s+1) &= \int_0^{\infty} t^s e^{-t} dt = -\int_0^{\infty} t^s d\left(e^{-t}\right) = -t^s e^{-t} \Big|_0^{\infty} + \int_0^{\infty} d(t^s)\, e^{-t} \\ &= s \int_0^{\infty} t^{s-1} e^{-t} dt = s\Gamma(s), \qquad s > 0. \end{aligned} \tag{3.7}$$

From repeated application of this functional equation we obtain for $n \in \mathbb{N}$,

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) \\ &\vdots \\ &= n(n-1)...2 \cdot 1\ \Gamma(1) = n! \end{aligned}$$

since $\Gamma(1) = \int_0^{\infty} e^{-t} dt = -e^{-t} \Big|_0^{\infty} = 1$.

Now we turn to finding series solutions for Bessel's equation (3.1) when 2ν is *not* an integer. Since the indicial roots are $\pm\nu$, we see that their difference

$\nu - (-\nu) = 2\nu$ is not an integer, and so there is a fundamental solution set of the form

$$\left\{ y_1 = x^{\nu} \sum_{n=0}^{\infty} c_n x^n, \quad y_2 = x^{-\nu} \sum_{n=0}^{\infty} d_n x^n \right\}, \qquad x > 0.$$

From (3.3) we have

$$\begin{aligned} 0 &= L\left[\sum_{n=0}^{\infty} c_n x^{n+r}\right] = \left(r^2 - \nu^2\right) c_0 x^r + \left((r+1)^2 - \nu^2\right) c_1 x^{1+r} \\ &\quad + \sum_{n=2}^{\infty} \left\{ \left[(n+r)^2 - \nu^2\right] c_n + c_{n-2} \right\} x^{n+r}, \end{aligned}$$

and hence with $r = \pm\nu$,

$$(n+r)^2 - \nu^2 = (n \pm \nu)^2 - \nu^2 = (n \pm \nu - \nu)(n \pm \nu + \nu) = n(n \pm 2\nu)$$

gives the recurrence relation

$$\begin{aligned} c_1 &= 0, \\ c_n &= -\frac{1}{n(n \pm 2\nu)} c_{n-2}, \qquad n \geq 2. \end{aligned}$$

Solving the recurrence gives

$$\begin{aligned} c_{2k+1} &= 0, \qquad k \geq 0, \\ c_{2k} &= \left(-\frac{1}{2k(2k \pm 2\nu)}\right)\left(-\frac{1}{(2k-2)(2k-2 \pm 2\nu)}\right) \dots \left(-\frac{1}{2(2 \pm 2\nu)}\right) c_0 \\ &= \left(-1^k\right) \frac{1}{2^{2k} k! (k \pm \nu) \dots (1 \pm \nu)} c_0 \\ &= \left(-1^k\right) \frac{\Gamma(1 \pm \nu)}{2^{2k} k! \Gamma(k \pm \nu + 1)} c_0, \end{aligned}$$

since by the functional equation (3.7),

$$\begin{aligned} \Gamma(k \pm \nu + 1) &= (k \pm \nu) \Gamma(k \pm \nu) = (k \pm \nu)(k \pm \nu - 1) \Gamma(k \pm \nu - 1) \\ &\vdots \\ &= (k \pm \nu)(k \pm \nu - 1) \dots (1 \pm \nu) \Gamma(1 \pm \nu). \end{aligned}$$

Thus we have

$$\begin{aligned} y_1(x) &= c_0 \sum_{n=0}^{\infty} \left(-1^k\right) \frac{\Gamma(1+\nu)}{2^{2k} k! \Gamma(k+\nu+1)} x^{2k+\nu}, \\ y_2(x) &= d_0 \sum_{n=0}^{\infty} \left(-1^k\right) \frac{\Gamma(1-\nu)}{2^{2k} k! \Gamma(k-\nu+1)} x^{2k-\nu}, \end{aligned}$$

for $x > 0$. It is customary to choose $c_0 = \frac{1}{2^{\nu}\Gamma(1+\nu)}$ and $d_0 = \frac{1}{2^{-\nu}\Gamma(1-\nu)}$ so that y_1 and y_2 become the Bessel functions of the first kind of orders ν and $-\nu$:

$$\begin{aligned} J_{\nu}(x) &= \sum_{k=0}^{\infty} \frac{\left(-1^k\right)}{k! \Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}, \\ J_{-\nu}(x) &= \sum_{k=0}^{\infty} \frac{\left(-1^k\right)}{k! \Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k-\nu}. \end{aligned}$$

Remark 7. *In the special case when* 2ν *is an integer, but* ν *is* not *an integer, i.e.* $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$*, it turns out that the series* J_ν *and* $J_{-\nu}$ *are linearly independent on* $(0,\infty)$*. Thus* $\{J_\nu, J_{-\nu}\}$ *is a fundamental solution set on* $(0,\infty)$ *for Bessel's equation (3.1) for all* ν *not an integer. When* ν *is a positive integer, it can be shown that* $J_{-\nu}$ *is a constant multiple of* J_ν*, and thus a second independent solution must involve a log term in this case.*

4. A caveat

The point $x = 0$ is a regular singular point of the equation

$$x\,(x-1)\,y'' + 3y' - 2y = 0.$$

We multiply the equation through by $\frac{x}{x-1}$ to obtain the Cauchy-Euler form

$$x^2 y'' + \left(\frac{3}{x-1}\right) xy' + \left(\frac{-2x}{x-1}\right) y = 0,$$

where $p\,(x) = \frac{3}{x-1}$ and $q\,(x) = \frac{-2x}{x-1}$ satisfy $p\,(0) = -3$ and $q\,(0) = 0$, to discover that the indicial polynomial is

$$P\,(\lambda) = \lambda\,(\lambda-1) - 3\lambda = \lambda^2 - 4\lambda = \lambda\,(\lambda-4)\,,$$

and has roots 0 and 4. We can then find the series solution corresponding to the larger root by substituting the series $y\,(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the equation and deriving the recurrence relation.

Here is the **caveat**! Since both $p\,(x)$ and $q\,(x)$ in the Cauchy-Euler form are *infinite* series, it will be easier to substitute $y\,(x)$ into the original equation, whose coefficients are simple polynomials. Thus we plug $y\,(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the original equation

$$0 = x^2 y'' - xy'' + 3y' - 2y,$$

and after some calculation get the recursion,

$$\begin{aligned} 0 &= -r\,(r-4)\,c_0, \\ c_{n+1} &= -\frac{(n+r)\,(n+r-1) - 2}{3\,(n+1+r) - (n+1+r)\,(n+r)} c_n \\ &= \frac{(n+r)\,(n+r-1) - 2}{(n+1+r)\,(n+r-3)} c_n, \qquad n \geq 0. \end{aligned}$$

After some more computation, we find that a nontrivial series solution corresponding to the larger root $r = 4$ is given by

$$y_1\,(x) = \sum_{n=0}^{\infty} (n+1)\,x^{n+4} = x^4 \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^{n+1}\right) = x^4 \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{x^4}{(1-x)^2},$$

and if we take the smaller root $r = 0$ in the recurrence relation we get

$$c_{n+1} = \frac{n\,(n-1) - 2}{(n+1)\,(n-3)} c_n, \qquad n \geq 0,$$

which runs aground when $n = 3$ since the fraction becomes infinite, $\frac{4}{4(0)}$, and forces $c_3 = c_2 = c_1 = c_0 = 0$. This then leaves c_4 unrestricted, but the recursion then simply recovers the known solution $y_1\,(x)$. Thus there is no second independent series solution, and by part (3) of Theorem 12, the second solution must have the form $y_2\,(x) = y_1\,(x) \ln x + \sum_{n=0}^{\infty} d_n x^n$.

CHAPTER 6

The Laplace transform

The Laplace transform $\mathcal{L}$ is mapping, or transform (we will define it in a moment), that takes certain functions $f(x)$ defined for x in $[0,\infty)$, into functions $F(s)$ defined for s in some semi-infinite interval (a,∞). We denote the transformed function $F(s)$ by $\mathcal{L}[f](s)$. The two main properties of this map are:

(1) $\mathcal{L}$ is linear, i.e. it transform sums to sums and scalar multiples to scalar multiples,

$$\mathcal{L}[c_1 f_1 + c_2 f_2](s) = c_1 \mathcal{L}[f_1](s) + c_2 \mathcal{L}[f_2](s),$$

(2) $\mathcal{L}$ transforms differentiation in x into multiplication by s, i.e.

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0).$$

The second property is clearly what makes the Laplace transform a valuable tool for solving linear differential equations, especially those with constant coefficients, as such equations are converted under the transform into *algebraic* equations. Unfortunately, the Laplace transform also converts multiplication by x into differentialtion by s, i.e.

$$\mathcal{L}[xf(x)](s) = -\frac{d}{ds}\mathcal{L}[f](s),$$

which limits its usefulness when applied to linear equations with variable coefficients. Moreover, we must also compute both the Laplace transform and its inverse, on familiar functions.

In order to state the definition of the Laplace transform $\mathcal{L}$, we require the notions of *piecewise continuity* and *growth of exponential order* for complex-valued functions $f:[0,\infty)\to\mathbb{C}$.

Definition 6. *We say f is* piecewise continuous *on $[0,\infty)$ if on each closed subinterval $[a,b]$ of $[0,\infty)$, there is a finite partition $\{a=t_0,t_1,...,t_N=b\}$ such that*

- *f is continuous on each open subinterval (t_{n-1},t_n), $1\le n\le N$,*
- *f has one-sided limits at each point in the partition, i.e.*

$$\lim_{x\to(t_n)^-} f(x) \text{ exists and } \lim_{x\to(t_n)^+} f(x) \text{ exists,}$$

for $1\le n\le N-1$ and the right hand limit exists at a and the left hand limit exists at b.

Definition 7. *We say $f(x)$ has exponential order c if there are constants $M<\infty$ and $T<\infty$ such that*

$$|f(x)| \le Me^{cx}, \qquad x>T.$$

DEFINITION 8. *If $f : [0, \infty) \to \mathbb{C}$ is piecewise continuous and of exponential order c, then we define the Laplace transform $\mathcal{L}[f](s)$ for $s > c$ by the improper integral*

$$\mathcal{L}[f](s) = \int_0^\infty f(x) e^{-sx} dx, \qquad s > c.$$

Note that for $f : [0, \infty) \to \mathbb{C}$ as in Definition 8, the integral $\int_0^T f(x) e^{-sx} dx$ exists since f is piecewise continuous, while the improper integral $\int_T^\infty f(x) e^{-sx} dx$ exists and converges absolutely for $s > c$ by the comparison test for integrals:

$$\begin{aligned}
\left| f(x) e^{-sx} \right| &\leq Me^{cx} e^{-sx} = Me^{(c-s)x}, \\
\int_T^\infty Me^{(c-s)x} dx &= \frac{M}{c-s} e^{(c-s)x} \mid_T^\infty \\
&= \frac{M}{c-s} \lim_{x\to\infty} e^{(c-s)x} - \frac{M}{c-s} e^{(c-s)T} \\
&= \frac{M}{s-c} e^{(c-s)T} < \infty, \qquad s > c.
\end{aligned}$$

Thus the Laplace transform $\mathcal{L}[f](s)$ is well-defined by the integral in Definition 8.

1. Properties of the Laplace transform

We begin by computing the Laplace transforms of some simple elementary functions:

$$\mathcal{L}[1](s) = \int_0^\infty 1 e^{-sx} dx = \frac{e^{-sx}}{-s} \mid_0^\infty = \frac{1}{s}, \qquad s > 0, \tag{1.1}$$

$$\begin{aligned}
\mathcal{L}[x](s) &= \int_0^\infty xe^{-sx} dx = x \frac{e^{-sx}}{-s} \mid_0^\infty - \int_0^\infty \frac{e^{-sx}}{-s} dx \\
&= \int_0^\infty \frac{e^{-sx}}{-s^2} dx = \frac{e^{-sx}}{-s^2} \mid_0^\infty = \frac{1}{s^2}, \qquad s > 0,
\end{aligned}$$

and for a complex,

$$\mathcal{L}[e^{ax}](s) = \int_0^\infty e^{ax} e^{-sx} dx = \frac{e^{(a-s)x}}{a-s} \mid_0^\infty = \frac{1}{s-a}, \qquad s > \operatorname{Re} a, \tag{1.2}$$

and for ω real,

$$\begin{aligned}
\mathcal{L}[\cos \omega x](s) &= \int_0^\infty \cos(\omega x) e^{-sx} dx = \int_0^\infty \frac{e^{i\omega x} + e^{-i\omega x}}{2} e^{-sx} dx \\
&= \frac{1}{2} \left\{ \mathcal{L}\left[e^{i\omega x}\right](s) + \mathcal{L}\left[e^{-i\omega x}\right](s) \right\} = \frac{1}{2} \left\{ \frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right\} \\
&= \frac{1}{2} \left\{ \frac{2s}{(s-i\omega)(s+i\omega)} \right\} = \frac{s}{s^2+\omega^2}, \qquad s > 0, \\
\mathcal{L}[\sin \omega x](s) &= \frac{\omega}{s^2+\omega^2}, \qquad s > 0.
\end{aligned}$$

Now we turn to a description and proof of the three main features of the Laplace transform $\mathcal{L}$, namely that

(1) $\mathcal{L}$ is linear,
(2) $\mathcal{L}$ interchanges differentiation and multiplication by the independent variable,

(3) $\mathcal{L}$ interchanges translation and multiplication by an exponential.

Here is a precise statement of these properties. When $f : [0, \infty) \to \mathbb{C}$ is piecewise continuous on $[0, \infty)$, the left hand limit at 0 exists, and we always assume $f(0)$ takes this value, so that f is continuous at 0. We define the unit step function $\mathcal{U}$ by

$$\mathcal{U}(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}.$$

THEOREM 13. *The following five properties hold for the Laplace transform* $\mathcal{L}[f](s) \equiv \int_0^\infty f(x) e^{-sx} dx$*:*

(1) *If* $f, g : [0, \infty) \to \mathbb{C}$ *are piecewise continuous and of exponential order* c*, and if* $\alpha, \beta \in \mathbb{C}$*, then*

$$\mathcal{L}[\alpha f + \beta g](s) = \alpha \mathcal{L}[f](s) + \beta \mathcal{L}[g](s), \qquad s > c.$$

(2) *If* $f, f', ..., f^{(n)} : [0, \infty) \to \mathbb{C}$ *are piecewise continuous and of exponential order* c*, then*

$$\mathcal{L}\left[f^{(n)}\right](s) = s^n \mathcal{L}[f](s) - \left\{f(0) s^{n-1} + f'(0) s^{n-2} + ... + f^{(n-1)}(0)\right\}, \qquad s > c.$$

(3) *If* $f : [0, \infty) \to \mathbb{C}$ *is piecewise continuous and of exponential order* c*, then* $x^n f(x)$ *is of exponential order* $c + \varepsilon$ *for all* $\varepsilon > 0$*, and*

$$\mathcal{L}[x^n f(x)](s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}[f](s), \qquad s > c.$$

(4) *Suppose* $a > 0$*. If* $f : [0, \infty) \to \mathbb{C}$ *is piecewise continuous and of exponential order* c*, then so is* $\mathcal{U}(x - a) f(x - a)$ *and*

$$\begin{aligned} \mathcal{L}[\mathcal{U}(x-a) f(x-a)](s) &= e^{-as} \mathcal{L}[f](s), \qquad s > c; \\ \mathcal{L}[\mathcal{U}(x-a) f(x)](s) &= e^{-as} \mathcal{L}[f(x+a)](s), \qquad s > c. \end{aligned}$$

(5) *If* $f : [0, \infty) \to \mathbb{C}$ *is piecewise continuous and of exponential order* c*, and if* $a \in \mathbb{R}$ *is any real number, then* $e^{ax} f(x)$ *is of exponential order* $c + a$ *and*

$$\mathcal{L}[e^{ax} f(x)](s) = \mathcal{L}[f](s - a), \qquad s > c + a.$$

Before proving the theorem, we give two simple illustrations of its application to solving an initial value problem for a constant coefficient nonhomogeneous linear equation. Consider the first order problem:

$$\begin{cases} y' - y &= 1 \\ y(0) &= 0 \end{cases}. \tag{1.3}$$

We know from earlier chapters that there is a unique solution to the initial value problem (1.3), and moreover that the solution is $y(x) = e^x - 1$. But if we merely assume that the solution and its derivative are piecewise continuous and of exponential order c, then we can apply the following five steps using the Laplace transform:

(1) Take the Laplace transform of both sides of $y' - y = 1$ to get

$$\mathcal{L}[y'] - \mathcal{L}[y] = \mathcal{L}[1].$$

(2) Denote the Laplace transform of y at s by $Y(s) = \mathcal{L}[y](s)$, and apply Theorem 13 to obtain

$$\{sY(s) - y(0)\} - Y(s) = \frac{1}{s}, \qquad s > c.$$

(3) Solve this *algebraic* equation for the transform $Y(s)$, and use the initial condition $y(0) = 0$, to obtain
$$Y(s) = \frac{1}{s(s-1)}.$$

(4) Use partial fractions to write
$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = \frac{-1}{s} + \frac{1}{s-1}.$$

(5) Finally, recognize from (1.1) and (1.2) used in *reverse*, that
$$\frac{-1}{s} = -\mathcal{L}[1] \text{ and } \frac{1}{s-1} = \mathcal{L}[e^x](s).$$

(6) Combining the previous five steps we have
$$\mathcal{L}[y](s) = Y(s) = \frac{1}{s(s-1)} = \frac{-1}{s} + \frac{1}{s-1} = \mathcal{L}[-1+e^x](s),$$
and now assuming uniqueness of Laplace transforms, i.e. that the Laplace transform is a one-to-one map, we conclude that $y = -1 + e^x$.

Now we use the same method to solve the more complicated initial value problem,

$$\left\{\begin{array}{ccc} y'' - 2y' + 5y & = & -8e^{-x} \\ y(0) & = & 2 \\ y'(0) & = & 12 \end{array}\right. . \tag{1.4}$$

We have
$$\left\{s^2 Y(s) - y(0)s - y'(0)\right\} - 2\left\{sY(s) - y(0)\right\} + 5Y(s) = -8\frac{1}{s+1},$$
from which we obtain
$$\begin{aligned} \left\{s^2 Y(s) - 2s - 12\right\} - 2\left\{sY(s) - 2\right\} + 5Y(s) &= -\frac{8}{s+1}; \\ \left\{s^2 - 2s + 5\right\} Y(s) &= 2s + 12 - 4 - \frac{8}{s+1}, \end{aligned}$$
and so
$$\begin{aligned} Y(s) &= \frac{1}{s^2 - 2s + 5}\left(2s + 8 - \frac{8}{s+1}\right) \\ &= \frac{(2s+8)(s+1) - 8}{(s^2 - 2s + 5)(s+1)} = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)}. \end{aligned}$$
Now the quadratic polynomial $s^2 - 2s + 5$ is irreducible, so we complete its square,
$$s^2 - 2s + 5 = (s-1)^2 + 2^2,$$
and write out the partial fraction decomposition of $Y(s)$ in terms of $s-1$ and 2:
$$Y(s) = \frac{2s^2 + 10s}{\left((s-1)^2 + 2^2\right)(s+1)} = \frac{A(s-1) + 2B}{(s-1)^2 + 2^2} + \frac{C}{s+1}.$$
Now
$$C = \frac{2s^2 + 10s}{(s-1)^2 + 2^2}\,|_{s=-1} = \frac{2-10}{\left((-1-1)^2 + 2^2\right)} = -1,$$

and so

$$A(s-1)+2B=\frac{2s^2+10s}{s+1}+\frac{(s-1)^2+2^2}{s+1}=\frac{3s^2+8s+5}{s+1}=3s+5,$$

which gives $A=3$ and $B=4$. Thus we have

$$Y(s)=3\frac{(s-1)}{(s-1)^2+2^2}+4\frac{2}{(s-1)^2+2^2}-\frac{1}{s+1}.$$

But from Theorem 13 we know

$$\begin{aligned}
\mathcal{L}[e^x\cos 2x](s) &= \frac{(s-1)}{(s-1)^2+2^2},\\
\mathcal{L}[e^x\sin 2x](s) &= \frac{2}{(s-1)^2+2^2},\\
\mathcal{L}\left[e^{-x}\right](s) &= \frac{1}{s+1},
\end{aligned}$$

and so from the uniqueness of Laplace transforms, the solution to the initial value problem (1.4) is

$$y=3e^x\cos 2x+4e^x\sin 2x-e^{-x}.$$

1.1. Proof of Theorem 13. Property (1) is an easy consequence of the linearity of convergent integrals. To prove property (2), we fix $s>c$ and integrate by parts to get

$$\begin{aligned}
\mathcal{L}\left[f^{(n)}\right](s) &= \int_0^\infty e^{-sx}f^{(n)}(x)\,dx=\int_0^\infty e^{-sx}d\left[f^{(n-1)}(x)\right]\\
&= e^{-sx}f^{(n-1)}(x)\,|_0^\infty = 0-f^{(n-1)}(0)-\int_0^\infty d\left[e^{-sx}\right]f^{(n-1)}(x)\\
&= -f^{(n-1)}(0)+\int_0^\infty se^{-sx}f^{(n-1)}(x)\\
&= -f^{(n-1)}(0)+s\mathcal{L}\left[f^{(n-1)}\right](s).
\end{aligned}$$

Now apply this identity repeatedly, or simply use induction on n, to obtain

$$\begin{aligned}
\mathcal{L}\left[f^{(n)}\right](s) &= -f^{(n-1)}(0)+s\mathcal{L}\left[f^{(n-1)}\right](s)\\
&= -f^{(n-1)}(0)+s\left\{-f^{(n-2)}(0)+s\mathcal{L}\left[f^{(n-2)}\right](s)\right\}\\
&= -f^{(n-1)}(0)-sf^{(n-2)}(0)+s^2\mathcal{L}\left[f^{(n-2)}\right](s)\\
&\vdots\\
&= -f^{(n-1)}(0)-sf^{(n-2)}(0)-...-s^{n-1}f(0)+s^n\mathcal{L}[f](s).
\end{aligned}$$

To prove (3), we first note that by L'Hôspital's rule, $\lim_{x\to\infty}\frac{x^n}{e^{\varepsilon x}}=0$, which shows that $x^nf(x)$ is of exponential order $c+\varepsilon$ for all $\varepsilon>0$. Then differentiating under the integral sign is justified, and yields

$$\frac{d}{ds}\mathcal{L}[f](s)=\frac{d}{ds}\int_0^\infty e^{-sx}f(x)\,dx=\int_0^\infty(-x)\,e^{-sx}f(x)\,dx=-\mathcal{L}[xf(x)](s).$$

Repeated application, or induction on n, then gives the formula

$$\frac{d^n}{ds^n}\mathcal{L}[f](s) = -\frac{d^{n-1}}{ds^{n-1}}\mathcal{L}[xf(x)](s) = \frac{d^{n-2}}{ds^{n-2}}\mathcal{L}\left[x^2 f(x)\right](s) = ... = (-1)^n \mathcal{L}[x^n f(x)](s).$$

Property (4) is a simple change of variable,

$$\begin{aligned}
\mathcal{L}[\mathcal{U}(x-a) f(x-a)](s) &= \int_0^\infty e^{-sx}\mathcal{U}(x-a) f(x-a)\,dx \\
&= \int_{-a}^\infty e^{-s(x+a)}\mathcal{U}((x+a)-a) f((x+a)-a)\,d(x+a) \\
&= \int_0^\infty e^{-sx}e^{-as} f(x)\,dx = e^{-as}\mathcal{L}[f](s); \\
\mathcal{L}[\mathcal{U}(x-a) f(x)](s) &= \int_a^\infty e^{-sx} f(x)\,dx \\
&= \int_0^\infty e^{-(s+a)x} f(x+a)\,dx = e^{-as}\mathcal{L}[f(x+a)](s),
\end{aligned}$$

and property (5) is just

$$\mathcal{L}[e^{ax} f(x)](s) = \int_0^\infty e^{-sx}e^{ax} f(x)\,dx = \int_0^\infty e^{-(s-a)x} f(x)\,dx = \mathcal{L}[f](s-a).$$

The Laplace transform is sometimes a convenient tool for solving nonhomogeneous intial value problems with piecewise continuous forcing functions.

Definition 9. *We write $\mathcal{L}^{-1}\{F(s)\}(x)$, or simply $\mathcal{L}^{-1}\{F(s)\}$, when $F(s) = \mathcal{L}[f(x)](s)$ is the Laplace transform of $f(x)$, and we refer to $\mathcal{L}^{-1}$ as the* inverse *Laplace transform.*

Example 33. *Solve the initial value problem*

$$\left\{\begin{array}{ccll} y'+y &=& f(x) & \text{for } x \neq \pi \\ y(0) &=& 5 & \end{array}\right. ;$$

$$\text{where } f(x) \equiv \left\{\begin{array}{lll} 20x & \text{for} & 0 \leq x < \pi \\ 3\cos x & \text{for} & x \geq \pi \end{array}\right. .$$

The forcing function $f(x)$ has a jump discontinuity at $x = \pi$, so we cannot require the differential equation to hold at $x = \pi$, but we do *require $y(x)$ to be continuous at π, and this uniquely determines the solution $y(x)$ to the initial value problem, as the solution on $[0, \infty]$ uniquely specifies the new initial condition $y(\pi)$ at $x = \pi$. To solve the initial value problem using the Laplace transform, we write the forcing function in terms of unit step functions for $x \geq 0$ as follows:*

$$\begin{aligned}
f(x) &= 20x - \mathcal{U}(x-\pi)\,20x + \mathcal{U}(x-\pi)\,3\cos x \\
&= 20x + \mathcal{U}(x-\pi)(3\cos x - 20x).
\end{aligned}$$

Then we compute

$$\begin{aligned}
\mathcal{L}[f](s) &= \mathcal{L}[20x](s) + \mathcal{L}[\mathcal{U}(x-\pi)(3\cos x - 20x)](s) \\
&= 20\frac{1}{s^2} + e^{-\pi s}\mathcal{L}[(3\cos(x+\pi) - 20(x+\pi))](s) \\
&= 20\frac{1}{s^2} + e^{-\pi s}\{-3\mathcal{L}[\cos x](s) - 20\mathcal{L}[x](s) - 20\pi\mathcal{L}[1](s)\} \\
&= 20\frac{1}{s^2} + e^{-\pi s}\left\{-3\frac{s}{s^2+1} - 20\frac{1}{s^2} - 20\pi\frac{1}{s}\right\}.
\end{aligned}$$

Taking the Laplace transform of the equation now gives

$$\begin{aligned}
sY(s) - y(0) + Y(s) &= \mathcal{L}[y'+y](s) = \mathcal{L}[f](s) \\
&= 20\frac{1}{s^2} + e^{-\pi s}\left\{-3\frac{s}{s^2+1} - 20\frac{1}{s^2} - 20\pi\frac{1}{s}\right\}; \\
(s+1)Y(s) &= 5 + 20\frac{1}{s^2} + e^{-\pi s}\left\{-3\frac{s}{s^2+1} - 20\frac{1}{s^2} - 20\pi\frac{1}{s}\right\}
\end{aligned}$$

and hence

$$\begin{aligned}
Y(s) &= \frac{5}{s+1} + 20\frac{1}{(s+1)s^2} \\
&\quad + e^{-\pi s}\left\{-3\frac{s}{(s+1)(s^2+1)} - 20\frac{1}{(s+1)s^2} - 20\pi\frac{1}{(s+1)s}\right\}.
\end{aligned}$$

Using partial fractions we get

$$\frac{s}{(s+1)s^2+1} = \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s^2+1} + \frac{\frac{1}{2}s}{s^2+1},$$

which gives the inverse transform

$$\begin{aligned}
&\mathcal{L}^{-1}\left[-3e^{-\pi s}\frac{s}{(s+1)(s^2+1)}\right] \\
&= -\frac{3}{2}\left\{-\mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{s^2+1}\right] + \mathcal{L}^{-1}\left[e^{-\pi s}\frac{s}{s^2+1}\right]\right\} \\
&= -\frac{3}{2}\left\{-\mathcal{U}(x-\pi)e^{-(x-\pi)} + \mathcal{U}(x-\pi)\sin(x-\pi) + \mathcal{U}(x-\pi)\cos(x-\pi)\right\}.
\end{aligned}$$

Similarly we obtain the inverse transforms

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{5}{s+1}\right] &= 5e^{-x}; \\
\mathcal{L}^{-1}\left[20\frac{1}{(s+1)s^2}\right] &= 20\mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{1-s}{s^2}\right] = 20e^{-x} + 20(x-1); \\
\mathcal{L}^{-1}\left[-20e^{-\pi s}\frac{1}{(s+1)s^2}\right] &= -20\mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{s+1} + e^{-\pi s}\frac{1-s}{s^2}\right] \\
&= -20\mathcal{U}(x-\pi)e^{-(x-\pi)} - 20\mathcal{U}(x-\pi)(x-\pi-1); \\
\mathcal{L}^{-1}\left[-20\pi e^{-\pi s}\frac{1}{(s+1)s}\right] &= -20\pi\mathcal{L}^{-1}\left[e^{-\pi s}\left(\frac{-1}{s+1} + \frac{1}{s}\right)\right] \\
&= 20\pi\mathcal{U}(x-\pi)e^{-(x-\pi)} - 20\pi\mathcal{U}(x-\pi).
\end{aligned}$$

Adding these all up we obtain

$$\begin{aligned} y(x) &= 5e^{-x}+20e^{-x}+20(x-1) \\ &\quad -\frac{3}{2}\left\{-\mathcal{U}(x-\pi)e^{-(x-\pi)}+\mathcal{U}(x-\pi)\sin(x-\pi)+\mathcal{U}(x-\pi)\cos(x-\pi)\right\} \\ &\quad -20\mathcal{U}(x-\pi)e^{-(x-\pi)}-20\mathcal{U}(x-\pi)(x-\pi-1) \\ &\quad +20\pi\mathcal{U}(x-\pi)e^{-(x-\pi)}-20\pi\mathcal{U}(x-\pi). \end{aligned}$$

We can of course collect all the terms multiplying the unit step function $\mathcal{U}(x-\pi)$ *to obtain the formulas*

$$\begin{aligned} y(x) &= 25e^{-x}+20x-20, \\ &\quad \textit{for } 0\le x\le\pi, \end{aligned}$$

and

$$\begin{aligned} y(x) &= 25e^{-x}-\frac{3}{2}\sin(x-\pi)-\frac{3}{2}\cos(x-\pi)+\left(20\pi-\frac{37}{2}\right)e^{-(x-\pi)}, \\ &\quad \textit{for } x\ge\pi. \end{aligned}$$

Note that $y(x)$ *is continuous at* $x=\pi$ *since both formulas give* $y(\pi)=25e^{-\pi}+20(\pi-1)$.

2. Convolutions and Laplace transforms

Consider the constant coefficient linear nonhomogeneous initial value problem with *vanishing* initial data:

$$\begin{aligned} &a_n y^{(n)}+a_{n-1}y^{(n-1)}+...+a_1y'+a_0y=f, \\ &y^{(n-1)}(0)=...=y'(0)=y(0)=0. \end{aligned}$$

The Laplace transform of this equation is

$$a_n s^n Y(s)+a_{n-1}s^{n-1}Y(s)+...+a_1 sY(s)+a_0Y(s)=F(s),$$

where $Y(s)=\mathcal{L}[y](s)$ and $F(s)=\mathcal{L}[f](s)$. We can factor the left hand side as $Y(s)$ times the characteristic polynomial

$$a_n s^n+a_{n-1}s^{n-1}+...+a_1 s+a_0=P(s).$$

Solving for $Y(s)$ we obtain

$$Y(s)=\frac{1}{P(s)}F(s),$$

which exhibits the Laplace transform $Y(s)$ of the solution $y(x)$ as a product of functions $\frac{1}{P(s)}$ and $F(s)$. Now it is often possible to find the inverse transforms of each of these functions separately, and the question that then arises is this:

- Given two functions $f(x)$ and $g(x)$ with Laplace transforms $F(s)$ and $G(s)$ respectively, what is the function $h(x)$ whose Laplace transform $H(s)$ is the product $F(s)G(s)$ of the transforms of $f(x)$ and $g(x)$? In other words, what is

$$\mathcal{L}^{-1}\{F(s)G(s)\}\ ?$$

To answer this question we calculate formally, without regard for rigor,

$$\begin{aligned} F(s)\,G(s) &= \left(\int_0^\infty e^{-sx} f(x)\,dx\right)\left(\int_0^\infty e^{-sy} g(y)\,dy\right) \\ &= \int_0^\infty \int_0^\infty e^{-sx} e^{-sy} f(x)\,g(y)\,dxdy. \end{aligned}$$

In the double integral, we make the change of variable

$$(x,y) \to (u,v)\,; \qquad u = x+y, v = y,$$

and using

$$\left|\det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\right| = \left|\det\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right| = 1,$$

we get

$$\begin{aligned} F(s)\,G(s) &= \int_{v=0}^{v=\infty}\left\{\int_{u=v}^{u=\infty} e^{-su} f(u-v)\,g(v)\left|\det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\right| du\right\} dv \\ &= \int_{u=0}^{u=\infty} e^{-su}\left\{\int_{v=0}^{v=u} f(u-v)\,g(v)\,dv\right\} du \\ &= \int_0^\infty e^{-su}\,(f*g)(u)\,du, \end{aligned}$$

where we have defined the *convolution* of f and g to be the function on $[0,\infty)$ given by the integral formula

$$(f*g)(u) = \int_0^u f(u-v)\,g(v)\,dv.$$

More formally, and using the usual dummy variables, we have this definition.

Definition 10. *Given f, g piecewise continuous on $[0,\infty)$, define their convolution $f*g$ on $[0,\infty)$ by*

$$(f*g)(x) = \int_0^x f(x-t)\,g(t)\,dt, \qquad x \geq 0.$$

Our formal calculation has thus shown that

$$F(s)\,G(s) = \int_0^\infty e^{-sx}\,(f*g)(x)\,dx = L[f*g](s)\,,$$

has inverse Laplace transform,

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = (f*g)(x)\,.$$

Note that the expression $f*g$ is linear in both f and g, and so may be thought of as a strange sort of *multiplication* of functions.

- The Laplace transform takes *convolution* multiplication of functions into ordinary *pointwise* multiplication of the transformed functions.

At this point it is useful to note that if both f and g are of exponential order, then so is $f*g$.

Lemma 6. *Suppose f and g are piecewise continuous and of exponential order c on $[0,\infty)$, i.e.*

$$|f(x)| \leq Me^{cx} \text{ and } |g(x)| \leq Me^{cx}, \qquad x > T.$$

Then $f * g$ is continuous and of exponential order $c + \varepsilon$ on $[0, \infty)$ for every $\varepsilon > 0$, i.e. there is a constant M_ε such that

$$|f * g(x)| \leq M_\varepsilon e^{(c+\varepsilon)x}, \qquad x \geq 0.$$

PROOF. We can take $T = 0$ at the expense of making M larger, i.e. there is a constant M' such that

$$|f(x)| \leq M' e^{cx} \text{ and } |g(x)| \leq M' e^{cx}, \qquad x \geq 0.$$

Then we have

$$\begin{aligned} |f * g(x)| &\leq \int_0^x |f(x-t)| \, |g(t)| \, dt \leq \int_0^x M' e^{c(x-t)} M' e^{ct} dt \\ &= (M')^2 \, x e^{cx} \leq M_\varepsilon e^{(c+\varepsilon)x}, \end{aligned}$$

for $x \geq 0$, since $\lim_{x\to\infty} \frac{x}{e^{\varepsilon x}} = \lim_{x\to\infty} \frac{1}{\varepsilon e^{\varepsilon x}} = 0$ by l'Hôspital's rule.

Finally, the continuity of $f * g$ at a point x follows by writing

$$\begin{aligned} & f * g(x+h) - f * g(x) \\ = & \int_0^{x+h} f(x+h-t) \, g(t) \, dt - \int_0^x f(x-t) \, g(t) \, dt \\ = & \int_0^{x+h} \{ f(x+h-t) - f(x-t) \} \, g(t) \, dt + \int_x^{x+h} f(x-t) \, g(t) \, dt \\ = & I(h) + II(h). \end{aligned}$$

Assume $0 < h \leq 1$ for convenience. Then

$$|II(h)| \leq \int_x^{x+h} |f(x-t) \, g(t)| \, dt \leq \int_x^{x+h} M' e^{c(x-t)} M' e^{ct} dt = (M')^2 \, e^{cx} h,$$

which goes to 0 as $h \to 0$. As for term $I(h)$, we have

$$\begin{aligned} |I(h)| &\leq \int_0^{x+h} |f(x+h-t) - f(x-t)| \, M' e^{ct} dt \\ &\leq M' e^{c(x+1)} \int_0^{x+1} |f(x+h-t) - f(x-t)| \, dt, \end{aligned}$$

and using the fact that f has only a *finite* number of jump discontinuities on the interval $[0, x+1]$, one can show (with some fuss) that

$$\lim_{h\to 0} \int_0^{x+1} |f(x+h-t) - f(x-t)| \, dt = 0.$$

Draw a picture to see that this conclusion is reasonable! $\square$

THEOREM 14 (The Convolution Theorem). *Suppose that f and g are piecewise continuous and of exponential order c on $[0, \infty)$. Then*

$$\mathcal{L}[f * g](s) = \mathcal{L}[f](s) \ \mathcal{L}[g](s), \qquad s > c.$$

PROOF. For $s > c$ we write the iterated integral for $\mathcal{L}[f * g](s)$ as a double integral, and make the substitution $u = x - t$, $v = t$ with Jacobian determinant 1, to get

$$\begin{aligned}
\mathcal{L}[f*g](s) &= \int_0^\infty e^{-sx}\left\{\int_0^x f(x-t)\,g(t)\,dt\right\}dx \\
&= \int_0^\infty \int_0^x e^{-s(x-t)} f(x-t)\,e^{-st} g(t)\,dtdx \\
&= \int_0^\infty \int_0^\infty e^{-su} f(u)\,e^{-sv} g(v)\,dvdu \\
&= \left(\int_0^\infty e^{-su} f(u)\,du\right)\left(\int_0^\infty e^{-sv} g(v)\,dv\right) \\
&= \mathcal{L}[f](s)\ \mathcal{L}[g](s).
\end{aligned}$$

□

EXAMPLE 34. *Here we use the Convolution Theorem to help compute the inverse Laplace transform*

$$\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+k^2}\right)^2\right\}, \qquad k \neq 0.$$

Indeed, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\}(x) = \frac{1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}(x) = \frac{1}{k}\sin kx \equiv f(x),$$

and so by the Convolution Theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+k^2}\right)^2\right\}(x) &= f*f(x) = \int_0^x f(x-t)\,f(t)\,dt \\
&= \int_0^x \left(\frac{1}{k}\sin k(x-t)\right)\left(\frac{1}{k}\sin kt\right)dt \\
&= \frac{1}{k^2}\int_0^x \frac{\cos k(x-2t)-\cos kx}{2}dt,
\end{aligned}$$

where in the last line we have used the trig identity

$$\sin A \sin B = \frac{\cos(A-B)-\cos(A+B)}{2}.$$

Continuing, we obtain

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+k^2}\right)^2\right\} &= \frac{1}{2k^2}\left\{\int_0^x \cos k(x-2t)\,dt - \int_0^x (\cos kx)\,dt\right\} \\
&= \frac{1}{2k^2}\frac{\sin k(x-2t)}{-2k}\ \Big|_0^x - \frac{1}{2k^2}x\cos kx \\
&= \frac{1}{2k^3}\{\sin kx - kx\cos kx\}.
\end{aligned}$$

2.1. Volterra integral equations. Now suppose that we are given continuous functions $g(x)$ and $h(x)$ on $[0,\infty)$, and consider the following *Volterra integral equation* for an unknown function $f(x)$:

$$f(x) = g(x) + \int_0^x h(x-t)\,f(t)\,dt = g(x) + h*f(x), \qquad x \geq 0. \tag{2.1}$$

This equation can be solved with the aid of the Laplace transform and the Convolution Theorem. Indeed, taking the Laplace transform we obtain

$$F(s) = G(s) + H(s) F(s),$$

where F, G, H are the Laplace transforms of f, g, h respectively. Thus we have

$$\begin{aligned} F(s) &= \frac{G(s)}{1 - H(s)}; \\ f(x) &= \mathcal{L}^{-1}\left\{\frac{G(s)}{1 - H(s)}\right\}(x), \qquad x \geq 0. \end{aligned}$$

EXAMPLE 35. *To solve the integral equation*

$$f(x) = 3x^2 - e^{-x} - \int_0^x e^{x-t} f(t)\, dt,$$

we must compute

$$f(x) = \mathcal{L}^{-1}\left\{\frac{G(s)}{1 - H(s)}\right\}(x),$$

where

$$g(x) = 3x^2 - e^{-x} \text{ and } h(x - t) = -e^{x-t}.$$

Thus we have

$$G(s) = 3\frac{2}{s^3} - \frac{1}{s+1} \text{ and } H(s) = -\frac{1}{s-1},$$

and so

$$\begin{aligned} \frac{G(s)}{1 - H(s)} &= \frac{1}{1 + \frac{1}{s-1}}\left(\frac{6}{s^3} - \frac{1}{s+1}\right) \\ &= \frac{s-1}{s}\frac{6s + 6 - s^3}{s^3(s+1)} = \frac{(s-1)\left(6s + 6 - s^3\right)}{s^4(s+1)}. \end{aligned}$$

The partial fraction decomposition is

$$\begin{aligned} \frac{G(s)}{1 - H(s)} &= \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} + \frac{E}{s+1} \\ &= \frac{-6}{s^4} + \frac{6}{s^3} + \frac{0}{s^2} + \frac{1}{s} + \frac{-2}{s+1}, \end{aligned}$$

and finally, taking inverse Laplace transforms yields

$$f(x) = -x^3 + 3x^2 + 1 - 2e^{-x}, \qquad x \geq 0.$$

2.2. A more general Volterra equation. The pair of functions $\{\cos x, \sin x\}$ is a fundamental solution set on the real line $\mathbb{R}$ for the homogeneous second order equation

$$y''(x) + y(x) = 0, \qquad x \in \mathbb{R},$$

and the general solution is given by

$$y_{\text{hom}}(x) = y_{\text{hom}}(0) \cos x + y'_{\text{hom}}(0) \sin x, \qquad x \in \mathbb{R}. \tag{2.2}$$

We now wish to solve the more general equation

$$y''(x) + y(x) = \sigma(x)\, y(x),$$

where σ is a continuous function on $\mathbb{R}$. First we solve the inhomogenoeous equation

$$y''(x) + y(x) = f(x)$$

by writing it as a system in $\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$:

$$\begin{aligned} \mathbf{y}' &= \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix} \\ &\equiv A\mathbf{y} + \mathbf{f}. \end{aligned}$$

Then the Wronskian matrix

$$W(x) = \begin{bmatrix} \cos x & \sin x \\ \cos' x & \sin' x \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

satisfies

$$W' = AW \text{ and } \left(W^{-1}\right)' = -W^{-1}A.$$

Thus

$$\begin{aligned} \left(W^{-1}\mathbf{y}\right)' &= W^{-1}\mathbf{y}' + \left(W^{-1}\right)'\mathbf{y} \\ &= W^{-1}\mathbf{y}' - W^{-1}A\mathbf{y} = W^{-1}\mathbf{f} \end{aligned}$$

implies

$$\mathbf{y} = W \int W^{-1}\mathbf{f}$$

and so a particular solution $y_{part}(x)$ is derived from

$$\begin{aligned} \begin{bmatrix} y_{part}(x) \\ y'_{part}(x) \end{bmatrix} &= \int_0^x W(x)\, W^{-1}(t)\, \mathbf{f}(t)\, dt \\ &= \int_0^x \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} dt \\ &= \int_0^x \begin{bmatrix} * & \sin x \cos t - \cos x \sin t \\ * & * \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} dt \\ &= \begin{bmatrix} \int_0^x \sin(x-t)\, f(t)\, dt \\ * \end{bmatrix}. \end{aligned} \tag{2.3}$$

Now we see from (2.2) and (2.3) that the solution to the initial value problem

$$\begin{cases} y'' + y &= \sigma y \\ y(0) &= 1 \\ y'(0) &= 0 \end{cases}$$

satisfies the integral equation

$$y(x) = \cos x + \int_0^x \sin(x-t)\, \sigma(t)\, y(t)\, dt, \qquad x \in \mathbb{R},$$

and vice versa. If we write $u(x) = \cos x$ and

$$Lh(x) = \int_0^x \sin(x-t)\, \sigma(t)\, h(t)\, dt = (\sin * \sigma h)(x),$$

we can rewrite this equation as

$$y = u + Ly, \tag{2.4}$$

an example of a more general type of Volterra integral equation than that considered in (2.1).

2.2.1. *Picard iterations.* To solve the Volterra equation (2.4) for $x \in [-N, N]$, we start with a guess $y_0 = y_0(x)$ where y_0 is any continuous function on $[-N, N]$, and plug it into the right side of (2.4), defining

$$\begin{aligned} y_1 &= y_1(x) = u(x) + Ly_0(x) \\ &= \cos x + \int_0^x \sin(x-t)\,\sigma(t)\,y_0(t)\,dt, \qquad x \in [-N, N]. \end{aligned}$$

If it happens that $y_1 = y_0$ (highly unlikely!) we are done. Otherwise set $y_2 = u + Ly_1$ and inductively

$$y_n = u + Ly_{n-1} \text{ on } [-N, N], \qquad n = 1, 2, 3, ... \tag{2.5}$$

We hope that this sequence of functions $\{y_n\}_{n=1}^{\infty}$ converges in some sense. Since uniform convergence yields a continuous limit, we define

$$\|h\| = \max_{|x| \le N} |h(x)|$$

and hope that $\|y_m - y_n\| \to 0$ as $m, n \to \infty$ (the Cauchy criterion for uniform convergence).

Now we compute inductively that

$$\begin{aligned} y_n &= u + Ly_{n-1} \\ &= u + L(u + Ly_{n-2}) \\ &\;\;\vdots \\ &= u + Lu + ... + L^{n-1}u + L^n y_0. \end{aligned} \tag{2.6}$$

Thus we have for $n > m$,

$$\begin{aligned} \|y_m - y_n\| &= \left\|L^m u + ... + L^{n-1}u + L^n y_0 - L^m y_0\right\| \\ &\le \|L^m u\| + ... + \left\|L^{n-1}u\right\| + \|L^n y_0\| + \|L^m y_0\|, \end{aligned} \tag{2.7}$$

and in particular this will tend to zero as $m, n \to \infty$ provided we have the "absolute convergence of orbit series":

$$\sum_{n=0}^{\infty} \|L^n v\| < \infty \text{ for every continuous } v \text{ on } [-N, N]. \tag{2.8}$$

Indeed, if (2.8) holds, then $\{y_n\}_{n=1}^{\infty}$ satisfies the Cauchy criterion for uniform convergence and hence there is a continuous function $y = y(x)$ on $[-N, N]$ such that $y_n \to y$ uniformly on $[-N, N]$. We now claim that y satisfies (2.4) on $[-N, N]$. For this we use the inequality

$$|Lv(x)| = \left|\int_0^x \sin(x-t)\,\sigma(t)\,v(t)\,dt\right| \le \|\sigma\|\,\|v\|\,|x|, \tag{2.9}$$

from which follows

$$\|Lv\| \le (N\|\sigma\|)\,\|v\| = C\,\|v\| \tag{2.10}$$

for all continuous v on $[-N, N]$. If we now let $n \to \infty$ in the equation (2.5) we obtain

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} (u + Ly_{n-1}) = u + Ly$$

since by (2.10),

$$\|Ly - Ly_{n-1}\| = \|L(y - y_{n-1})\| \le C\,\|y - y_{n-1}\| \to 0, \qquad \text{as } n \to \infty.$$

Finally we establish the "absolute convergence of orbit series" in (2.8). By (2.9) we have

$$\left|\int_0^x \sin(x-t)\,\sigma(t)\,Lv(t)\,dt\right| \le \int_0^x |\sin(x-t)\,\sigma(t)|\left\{\|\sigma\|\,\|v\|\,|t|\right\}dt \le \|\sigma\|^2\,\|v\|\,\frac{|x|^2}{2},$$

and continuing by induction we obtain

$$\begin{aligned}|L^n v(x)| &= \left|\int_0^x \sin(x-t)\,\sigma(t)\,L^{n-1}v(t)\,dt\right| \le \|\sigma\|^n\,\|v\|\,\frac{|x|^n}{n!},\\ \|L^n v\| &\le \|\sigma\|^n\,\|v\|\,\frac{N^n}{n!},\end{aligned}$$

from which (2.8) follows immediately:

$$\sum_{n=0}^{\infty}\|L^n v\| \le \sum_{n=0}^{\infty}\|\sigma\|^n\,\|v\|\,\frac{N^n}{n!} = e^{N\|\sigma\|}\,\|v\| < \infty.$$

3. Transforms of integrals, periodic functions and the delta function

If we set $f\equiv 1$ in the Convolution Theorem, we get

$$\mathcal{L}[1*g](s) = \mathcal{L}[1](s)\;\mathcal{L}[g](s) = \frac{1}{s}\mathcal{L}[g](s),$$

where

$$1*g(x) = \int_0^x g(t)\,dt$$

is the antiderivative of g that vanishes at the origin. Thus with $F(s)=\mathcal{L}[f](s)$ we have the formulas

$$\mathcal{L}\left[\int_0^x f(t)\,dt\right](s) = \frac{F(s)}{s} \text{ and } \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}(x) = \int_0^x f(t)\,dt. \tag{3.1}$$

Now suppose that $f(x)$ is *periodic* with period T on $[0,\infty)$, i.e.

$$f(x+T) = f(x), \qquad x\ge 0,$$

as well as being piecewise continuous and of exponential order c. Then the Laplace transform is given by the following integral over the initial period:

$$\mathcal{L}[f(x)](s) = \frac{1}{1-e^{-sT}}\int_0^T e^{-sx}f(x)\,dx, \qquad s>c. \tag{3.2}$$

Indeed, using the substitution $x\to x+T$ in the second integral below, followed by the periodicity of f, we have

$$\begin{aligned}\mathcal{L}[f(x)](s) &= \int_0^T e^{-sx}f(x)\,dx + \int_T^\infty e^{-sx}f(x)\,dx\\ &= \int_0^T e^{-sx}f(x)\,dx + \int_0^\infty e^{-s(x+T)}f(x+T)\,dx\\ &= \int_0^T e^{-sx}f(x)\,dx + \int_0^\infty e^{-s(x+T)}f(x)\,dx\\ &= \int_0^T e^{-sx}f(x)\,dx + e^{-sT}\mathcal{L}[f(x)](s).\end{aligned}$$

Solving for $\mathcal{L}[f(x)](s)$ gives (3.2).

Finally we consider the simplest of the "generalized functions" or "distributions", namely the 'Dirac delta function with pole at x_0' $\delta(x-x_0)$, where x_0 is any fixed real number, and x is the independent variable. This generalized function is assumed to have the following two properties:

(1) $\delta(x-x_0)=0$ for all $x\neq x_0$,
(2) $\int_{-\infty}^{\infty}\delta(x-x_0)\,dx=1$ for all fixed x_0.

Of course there is no actual function with these apparently contradictory properties, but the following sequence $\{f_n\}_{n=1}^{\infty}$ of step functions has these properties 'in the limit':

$$f_n(x)\equiv n\mathbf{1}_{\left[x_0,x_0+\frac{1}{n}\right]}(x),\qquad x\in\mathbb{R},n\in\mathbb{N},$$

where $\mathbf{1}_{\left[x_0,x_0+\frac{1}{n}\right]}=\mathcal{U}(x-x_0)-\mathcal{U}\left(x-x_0-\frac{1}{n}\right)$ is the indicator function of the interval $\left[x_0,x_0+\frac{1}{n}\right]$. So also does the sequence of continuous functions

$$f_n(x)\equiv\frac{\pi n}{2}\sin(\pi nx)\,\mathbf{1}_{\left[0,\frac{1}{n}\right]}(x),\qquad x\in\mathbb{R},n\in\mathbb{N},$$

as well as many other sequences of functions whose supports shrink to x_0 and whose integrals tend to 1 as $n\to\infty$. It is for this reason that the Dirac delta function $\delta(x-x_0)$ is referred to as a 'generalized' function.

The Laplace transform of $\delta(x-x_0)$ can be taken to be the limit of the Laplace transform of any such sequence, and choosing the sequence of step functions $\{f_n\}_{n=1}^{\infty}$ above, we get

$$\begin{aligned}\mathcal{L}[\delta(x-x_0)](s) &= \lim_{n\to\infty}\mathcal{L}[f_n(x)](s)=\lim_{n\to\infty}n\left\{\mathcal{L}[\mathcal{U}(x-x_0)](s)-\mathcal{L}\left[\mathcal{U}\left(x-x_0-\frac{1}{n}\right)\right]\right\}\\ &= \lim_{n\to\infty}n\left\{\frac{e^{-sx_0}}{s}-\frac{e^{-s\left(x_0+\frac{1}{n}\right)}}{s}\right\}=-\frac{1}{s}\lim_{n\to\infty}\frac{e^{-s\left(x_0+\frac{1}{n}\right)}-e^{-sx_0}}{\frac{1}{n}}\\ &= -\frac{1}{s}\left(\frac{d}{dx}e^{-sx}\right)|_{x=x_0}=e^{-sx_0},\end{aligned}$$

and hence the formulas

$$\mathcal{L}[\delta(x-a)](s)=e^{-sa}\text{ and }\mathcal{L}^{-1}\left\{e^{-as}\right\}(x)=\delta(x-a).\tag{3.3}$$

Example 36. *The initial value problem*

$$\left\{\begin{array}{ccc}2y''+y'+2y&=&\delta(x-5)\\ y(0)&=&0\\ y'(0)&=&0\end{array}\right.,$$

has a forcing function that is a unit impulse at $x=5$. To solve this problem, we take the Laplace transform to obtain

$$\begin{aligned}&\left(2s^2+s+2\right)Y(s)=e^{-5s};\\ &Y(s)=\frac{e^{-5s}}{2s^2+s+2}=\frac{e^{-5s}}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2};\\ &y(x)=\frac{2}{\sqrt{15}}\mathcal{U}(x-5)\,e^{-\frac{x-5}{4}}\sin\frac{\sqrt{15}}{4}(x-5),\end{aligned}$$

since $\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right\}=\frac{4}{\sqrt{15}}e^{-\frac{x}{4}}\sin\frac{\sqrt{15}}{4}x.$

CHAPTER 7

First order systems

Recall that in Chapter 4, in connection with higher order equations, we introduced the *general* first order system (1.1) of n equations in n unknown functions, $\{y_1, y_2, ..., y_n\}$,

$$\left\{\begin{array}{ccc} y_1' & = & f_1\left(x, y_1, y_2, ..., y_{n-1}\right) \\ y_2' & = & f_2\left(x, y_1, y_2, ..., y_{n-1}\right) \\ \vdots & \vdots & \vdots \\ y_{n-1}' & = & f_{n-1}\left(x, y_1, y_2, ..., y_{n-1}\right) \\ y_n' & = & f_n\left(x, y_1, y_2, ..., y_{n-1}\right) \end{array}\right. ,$$

where the functions $f_k\left(x, y_1, y_2, ..., y_{n-1}\right)$ are typically arbitrary for $k = 1, 2, ..., n-1, n$. The system (1.2) can be written more profitably in vector form

$$\mathbf{y}' = \mathbf{f}\left(x, \mathbf{y}\right),$$

where we use boldface type to denote n-dimensional vectors,

$$\begin{aligned} \mathbf{y} &= \left(y_1, y_2, ..., y_n\right) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \\ \mathbf{f}\left(x, \mathbf{y}\right) &= \left(f_1\left(x, \mathbf{y}\right), f_2\left(x, \mathbf{y}\right), ..., f_n\left(x, \mathbf{y}\right)\right) = \begin{pmatrix} f_1\left(x, \mathbf{y}\right) \\ f_2\left(x, \mathbf{y}\right) \\ \vdots \\ f_n\left(x, \mathbf{y}\right) \end{pmatrix}, \end{aligned}$$

which we write as either row vectors or column vectors depending on context. In our Existence and Uniqueness Theorem, we showed that if

- $\mathcal{R}$ is an open region of the Euclidean space $\mathbb{R}^{n+1}$,
- if $\mathbf{f} : \mathcal{R} \to \mathbb{R}^n$ is continuous,
- if $\mathbf{P}_0 = \left(x_0, \mathbf{y}_0\right) \in \mathcal{R}$,
- and if $\mathbf{f}\left(x, \mathbf{y}\right)$ satisfies a *Lipschitz* condition in $\mathcal{R}$ in the $\mathbf{y}$ variables,

then the $n \times n$ initial value problem

$$\left\{\begin{array}{ccc} \mathbf{y}' & = & \mathbf{f}\left(x, \mathbf{y}\right) \\ \mathbf{y}\left(x_0\right) & = & \mathbf{y}_0 \end{array}\right. ,$$

has a unique solution defined in some open interval containing x_0.

This 'system' point of view proved useful not only in establishing existence and uniqueness for higher order equations, but also in

(1) characterizing fundamental solution sets to *homogeneous* linear equations in terms of the Wronskian via Abel's formula,

(2) deriving the method of variation of parameters for solving higher order *nonhomogeneous* linear variable coefficient equations,
(3) and proving the existence of power series solutions for linear equations when the coefficients are analytic.

It turns out that we can easily establish analogues of Abel's formula and variation of parameters for $n \times n$ systems, to which we now turn.

1. Abel's formula and variation of parameters

Recall that in Chapter 5, we introduced the first order *linear* system of n equations in n unknown functions $\{y_1, y_2, ..., y_n\}$,

$$\left\{ \begin{array}{ccc} y_1' & = & a_{1,1}(x)\, y_1 + ... a_{1,n}(x)\, y_n \\ y_2' & = & a_{2,1}(x)\, y_1 + ... a_{2,n}(x)\, y_n \\ \vdots & \vdots & \vdots \\ y_{n-1}' & = & a_{n-1,1}(x)\, y_1 + ... a_{n-1,n}(x)\, y_n \\ y_n' & = & a_{n,1}(x)\, y_1 + ... a_{n,n}(x)\, y_n \end{array} \right. .$$

which with

$$A(x) \equiv \left[\begin{array}{cccc} a_{1,1}(x) & a_{1,2}(x) & \cdots & a_{1,n}(x) \\ a_{2,1}(x) & a_{2,2}(x) & \cdots & a_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n}(x) \end{array} \right], \mathbf{y}(x) \equiv \left(\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \right),$$

we rewrote succinctly as

$$\mathbf{y}'(x) = A(x)\, \mathbf{y}(x) . \tag{1.1}$$

A fundamental solution set (of column vector solutions)

$$\Phi \equiv \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n\} = \left\{ \left(\begin{array}{c} y_{1,1} \\ y_{2,1} \\ \vdots \\ y_{n,1} \end{array} \right), \left(\begin{array}{c} y_{1,2} \\ y_{2,2} \\ \vdots \\ y_{n,2} \end{array} \right), ..., \left(\begin{array}{c} y_{1,n} \\ y_{2,n} \\ \vdots \\ y_{n,n} \end{array} \right) \right\},$$

can be written in the form of a matrix with columns $\mathbf{y}_k$,

$$\Phi(x) \equiv \left[\begin{array}{cccc} y_{1,1}(x) & y_{1,2}(x) & \cdots & y_{1,n}(x) \\ y_{2,1}(x) & y_{2,2}(x) & \cdots & y_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1}(x) & y_{n,2}(x) & \cdots & y_{n,n}(x) \end{array} \right]$$

and satisfies the matrix equation,

$$\begin{array}{rcl} \Phi'(x) & = & A(x)\, \Phi(x), \\ \det \Phi(x) & \neq & 0. \end{array}$$

The fundamental solution set Φ is *normalized* at x_0 if the matrix $\Phi(x_0)$ is the identity matrix.

Recall that the *trace* of a matrix $A = [a_{ij}]_{i,j=1}^n$ is defined to be the sum of the diagonal elements,

$$\textit{trace}\ A \equiv \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + ... + a_{nn}.$$

THEOREM 15 (Abel's formula). *If Φ is a fundamental solution set for the linear $n \times n$ system (1.1), then*

$$\det \Phi(x) = \det \Phi(x_0)\, e^{\int_{x_0}^{x} [trace\ A(t)]dt}.$$

PROOF. We compute using the product rule and the equation (1.1),

$$\begin{aligned}
\frac{d}{dx} \det \Phi(x) &= \frac{d}{dx} \begin{bmatrix} y_{1,1}(x) & y_{1,2}(x) & \cdots & y_{1,n}(x) \\ y_{2,1}(x) & y_{2,2}(x) & \cdots & y_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1}(x) & y_{n,2}(x) & \cdots & y_{n,n}(x) \end{bmatrix} \\
&= \sum_{i=1}^{n} \det \begin{bmatrix} y_{1,1}(x) & y_{1,2}(x) & \cdots & y_{1,n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ y'_{i,1}(x) & y'_{i,2}(x) & \cdots & y'_{i,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1}(x) & y_{n,2}(x) & \cdots & y_{n,n}(x) \end{bmatrix} \\
&= \sum_{i=1}^{n} \det \begin{bmatrix} y_{1,1}(x) & y_{1,2}(x) & \cdots & y_{1,n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=1}^{n} a_{i,k}(x)\, y_{k,1}(x) & \sum_{k=1}^{n} a_{i,k}(x)\, y_{k,2}(x) & \cdots & \sum_{k=1}^{n} a_{i,k}(x)\, y_{k,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1}(x) & y_{n,2}(x) & \cdots & y_{n,n}(x) \end{bmatrix}.
\end{aligned}$$

Now using the multilinear and alternating properties of determinants, we get that only the case $k = i$ in the sum survives to give

$$\begin{aligned}
\frac{d}{dx} \det \Phi(x) &= \sum_{i=1}^{n} \det \begin{bmatrix} y_{1,1}(x) & y_{1,2}(x) & \cdots & y_{1,n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ a_{i,i} y_{i,1}(x) & a_{i,i} y_{i,2}(x) & \cdots & a_{i,i}(x)\, y_{i,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1}(x) & y_{n,2}(x) & \cdots & y_{n,n}(x) \end{bmatrix} \\
&= \sum_{i=1}^{n} a_{i,i}(x) \det \Phi(x) = [trace\ A(x)] \det \Phi(x).
\end{aligned}$$

Solving this scalar equation for $\det \Phi(x)$, we obtain

$$\ln |\det \Phi(x)| - \ln |\det \Phi(x_0)| = \int_{x_0}^{x} \frac{d}{dt} \ln |\det \Phi(t)|\, dt = \int_{x_0}^{x} [trace\ A(t)]\, dt,$$

which gives Abel's formula. $\square$

Now we derive the method of *variation of parameters* for solving the *nonhomogeneous* equation

$$\mathbf{y}'(x) = A(x)\, \mathbf{y}(x) + \mathbf{f}(x). \tag{1.2}$$

Given a fundamental solution set $\Phi = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n\}$ to the homogeneous equation (1.1), we substitute the vector

$$\begin{aligned}
\mathbf{y}_p(x) &\equiv v_1(x)\, \mathbf{y}_1(x) + ... + v_n(x)\, \mathbf{y}_n(x) \\
&= \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n\}\, \mathbf{v}(x) = \Phi(x)\, \mathbf{v}(x),
\end{aligned}$$

into the nonhomogeneous equation (1.2) to get

$$\begin{aligned} A(x)\,\Phi(x)\,\mathbf{v}(x)+\mathbf{f}(x) &= A(x)\,\mathbf{y}_p(x)+\mathbf{f}(x) \\ &= \mathbf{y}_p'(x)=\Phi'(x)\,\mathbf{v}(x)+\Phi(x)\,\mathbf{v}'(x)\,. \end{aligned}$$

Now $\Phi' = A\Phi$ since Φ is a fundamental solution set, and so we have the following first order equation for the vector $\mathbf{v}(x)$:

$$\Phi(x)\,\mathbf{v}'(x)=\mathbf{f}(x)\,.$$

But $\det\Phi \neq 0$ as well since Φ is a fundamental solution set, and so $\Phi(x)$ is invertible and we have

$$\begin{aligned} \mathbf{v}'(x) &= \Phi(x)^{-1}\,\mathbf{f}(x)\,; \\ \mathbf{v}(x) &= \int \Phi(t)^{-1}\,\mathbf{f}(t)\,dt. \end{aligned}$$

This gives a particular solution

$$\mathbf{y}_p(x)=\Phi(x)\,\mathbf{v}(x)=\Phi(x)\int \Phi(t)^{-1}\,\mathbf{f}(t)\,dt,$$

and because of the constant vector of integration, we actually get the general solution from this formula. More precisely we have

$$\begin{aligned} \mathbf{y}(x) &= \Phi(x)\left(\int_{x_0}^{x}\Phi(t)^{-1}\,\mathbf{f}(t)\,dt+\mathbf{c}\right) \\ &= \left\{\Phi(x)\int_{x_0}^{x}\Phi(t)^{-1}\,\mathbf{f}(t)\,dt\right\}+\{c_1\mathbf{y}_1(x)+\dots+c_n\mathbf{y}_n(x)\} \\ &\equiv \mathbf{y}_p(x)+\mathbf{y}_c(x)\,, \end{aligned}$$

where

$$\mathbf{y}_p(x)=\Phi(x)\int_{x_0}^{x}\Phi(t)^{-1}\,\mathbf{f}(t)\,dt$$

is a *particular* solution to the nonhomogeneous equation (1.2), and

$$\mathbf{y}_c(x)=c_1\mathbf{y}_1(x)+\dots+c_n\mathbf{y}_n(x)$$

is the complementary solution, i.e. the general solution to the *homogeneous* equation (1.1).

2. Constant coefficient linear systems

The method of variation of parameters reduces the solution of the nonhomogeneous linear variable coefficient equation (1.2) to the problem of finding a fundamental solution set for the corresponding homogeneous equation (1.1). In general we cannot find explicit elementary solutions to the homogeneous system (1.1). Recall however, that in Theorem 8, we found explicit elementary solutions for the scalar equation $L[y]=0$ when L is constant coefficient n^{th} order linear differential operator as in (3.1). It turns out that we are also able to find an explicit fundamental solution set for the system (1.1) when the matrix $A(x)=A\equiv\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}$ is *constant*. We now describe the details.

Suppose that A is a constant $n \times n$ matrix and consider the homogeneous linear system

$$\mathbf{y}'(x) = A\mathbf{y}(x). \tag{2.1}$$

Our strategy here is similar to that used several times previously, namely we devise a proof in the scalar case $n = 1$, and generalize it to work for $n \times n$ matrices. The scalar case is the simple equation

$$y'(x) = ay(x), \tag{2.2}$$

where a is a 1×1 matrix, i.e. a number, and this equation has general solution

$$y(x) = ce^{ax}, \qquad x \in \mathbb{R},\ c \in \mathbb{R}.$$

Now we note that the solution $y = e^{ax}$ has a power series expansion,

$$y = \sum_{n=0}^{\infty} \frac{a^n}{n!} x^n, \qquad x \in \mathbb{R},$$

and most importantly, that *from this power series expansion alone*, we can deduce that it is a solution to the equation (2.2):

$$\begin{aligned}
y(x)' &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{n!} a^n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \frac{d}{dx} x^n = \sum_{n=1}^{\infty} \frac{1}{n!} a^n n x^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} a^{n+1} (n+1) x^n = a \sum_{n=0}^{\infty} \frac{1}{n!} a^n x^n = ay(x),
\end{aligned}$$

since $\frac{1}{(n+1)!} a^{n+1} (n+1) = a \frac{1}{(n+1)n!} a^n (n+1) = a \frac{1}{n!} a^n$.

This observation suggests that the same procedure may work for higher order matrices. So given an $n \times n$ matrix A we consider the matrix-valued power series $\Phi(x)$ with matrix coefficients given by,

$$\Phi(x) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} A^n x^n,$$

and where by convention, we define A^0 to be the $n \times n$ identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

with 1's down the main diagonal, and 0's elsewhere. The series above has infinite radius of convergence, and so converges absolutely for all $x \in \mathbb{R}$. Moreover the following calculations are valid within the open interval of convergence $\mathbb{R}$:

$$\begin{aligned}
\Phi'(x) &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{n!} A^n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \frac{d}{dx} x^n = \sum_{n=1}^{\infty} \frac{1}{n!} A^n n x^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} A^{n+1} (n+1) x^n = A \sum_{n=0}^{\infty} \frac{1}{n!} A^n x^n = A\Phi(x).
\end{aligned}$$

Note how these matrix calulations exactly mirror those with numbers done above! We have thus obtained a matrix solution $\Phi(x)$ to the equation $\Phi'(x) = A\Phi(x)$, and if we list the columns of the matrix solution $\Phi(x)$ as $\mathbf{y}_1(x), \ldots \mathbf{y}_n(x)$, then each vector function $\mathbf{y}_j(x)$ is a solution to the system (2.1), i.e. $\mathbf{y}_j'(x) = A\mathbf{y}_j(x)$ for $j = 1, 2, ..., n$. We conclude that

$$\Phi = \{\mathbf{y}_1, ...\mathbf{y}_n\}$$

will be a fundamental solution set for (2.1) provided $\det \Phi(x) \neq 0$. But $\det \Phi(0) = \det I = 1$, and Abel's formula now shows that

$$\det \Phi(x) = \det \Phi(0)\, e^{\int_0^x [trace\ A(t)]dt} = e^{\int_0^x [trace\ A(t)]dt} \neq 0, \qquad x \in \mathbb{R}.$$

Thus we have shown that

$$\{\mathbf{y}_1, ...\mathbf{y}_n\}$$

is a fundamental solution set for the system (2.1), where the vectors $\mathbf{y}_j$ are the columns of the matrix solution

$$\Phi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n x^n.$$

At this point we define the *exponential* of an $n \times n$ matrix B, and which we denote by e^B, by the series

$$e^B \equiv \sum_{n=0}^{\infty} \frac{1}{n!} B^n.$$

With this notation our matrix solution becomes $\Phi(x) = e^{Ax}$ simply because $(Ax)^n = A^n x^n$ for all $n \geq 0$. In fact the general solution to the system (2.1) is now seen to be given by

$$\mathbf{y}(x) = e^{Ax}\mathbf{c}, \qquad x \in \mathbb{R},$$

where $\mathbf{c}$ is an arbitrary n-vector. The interpretation of $\mathbf{c}$ is that it is the initial condition satisfied by the solution $\mathbf{y}(x)$,

$$\mathbf{y}(0) = e^{A0}\mathbf{c} = I\mathbf{c} = \mathbf{c}.$$

2.1. Calculation of the exponential of a matrix. The only problem remaining with our general solution $\mathbf{y}(x) = e^{Ax}\mathbf{c}$ to the system (2.1) is the problem of computing the matrix series e^{Ax} so as to identify the components as elementary functions. For this we turn first to the relatively uncomplicated case $n = 2$. Given the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we wish to compute

$$e^{Ax} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^n x^n.$$

This is relatively easy in the special case when A is a diagonal matrix $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, since then we have

$$A^n = \begin{bmatrix} a^n & 0 \\ 0 & d^n \end{bmatrix}, \qquad n \geq 0,$$

which is easily proved by induction on n. Plugging this into the series gives us

$$\begin{aligned} e^{Ax} &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} a^n & 0 \\ 0 & d^n \end{bmatrix} x^n = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{1}{n!}a^n x^n & 0 \\ 0 & \frac{1}{n!}d^n x^n \end{bmatrix} \\ &= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}a^n x^n & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!}d^n x^n \end{bmatrix} = \begin{bmatrix} e^{ax} & 0 \\ 0 & e^{dx} \end{bmatrix}. \end{aligned}$$

It is now easy to see that for square diagonal matrices of *any* size, we compute the exponential matrix just by replacing each entry on the diagonal by its ordinary exponential!

Now we invoke some trickery from linear algebra. For this we return for the moment to $n \times n$ matrices and recall the notions of *eigenvalues* and *eigenvectors* of an $n \times n$ matrix A. An eigenvalue/eigenvector pair $(\lambda, \mathbf{v})$ for A consists of a *nonzero* vector $\mathbf{v}$ that is mapped into a multiple of itself with magnification factor λ, i.e.

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{2.3}$$

Note that such a vector is determined by its direction, since any multiple of it will also satisfy (2.3). For example, a diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

has eigenvectors $\mathbf{e}_j$ with corresponding eigenvalues a_{jj}. Thus $(\lambda_j, \mathbf{v}_j) = (a_{jj}, \mathbf{e}_j)$ is an eigenpair for A for each $1 \leq j \leq n$.

Recall also that the eigenvalues can be found by calculating the roots of the *characteristic polynomial* of A:

$$P(\lambda) = \det(\lambda I - A).$$

The reason for this is that the following statements are equivalent:

(1) λ is an eigenvalue for A,
(2) there is a *nonzero* vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$,
(3) $(\lambda I - A)\mathbf{v} = \lambda\mathbf{v} - A\mathbf{v} = 0$ for some $\mathbf{v} \neq 0$,
(4) $\lambda I - A$ is not an invertible matrix,
(5) $\det(\lambda I - A) = 0$.

Thus we see in particular that there are at most n eigenvalues. If a root λ of $P(\lambda)$ is repeated m times, we say that λ is an eigenvalue of mulitplicity m of A. Corresponding eigenvectors $\mathbf{v}$ for λ can be found by solving the matrix equation

$$(\lambda I - A)\mathbf{v} = \mathbf{0}.$$

If λ is an eigenvalue of mulitiplicity m, it can be shown that the vector space of solutions to this equation, i.e. the space of corresponding eigenfunctions, has dimension at least 1, and at most m, but not necessarily equal to m. In particular, if λ has mulitiplicity 1, there is a *unique* eigenvector.

EXAMPLE 37. *The matrix* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *has characteristic polynomial* $P(\lambda) = (\lambda - 1)^2$, *so has the single eigenvalue* 1 *of multiplicity* 2. *However, it is easy to see that* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *(or any multiple) is the only eigenvector.*

We will see below that the exponential e^A of a matrix A has an especially simple form when A has n *linearly independent eigenvectors* $\mathbf{v}_1, ..., \mathbf{v}_n$. This condition on a matrix A is easily seen to be equivalent to the requirement that A is *diagonalizable*, i.e. there is an invertible matrix B such that

$$B^{-1}AB = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

is a diagonal matrix. Indeed, we can take B to be the matrix with columns $\mathbf{v}_1, ..., \mathbf{v}_n$ so that B takes $\mathbf{e}_j$ to $\mathbf{v}_j$ for $1 \leq j \leq n$. Then $B^{-1}AB$ takes $\mathbf{e}_j$ to

$$B^{-1}AB\mathbf{e}_j = B^{-1}A\mathbf{v}_j = B^{-1}\lambda_j\mathbf{v}_j = \lambda_j B^{-1}\mathbf{v}_j = \lambda_j\mathbf{e}_j,$$

and hence is the diagonal matrix Λ with diagonal entries $\{\lambda_1, ..., \lambda_n\}$.

We also observe that the eigenvalues of A are precisely the diagonal entries of Λ, counted according to multiplicity. Indeed both A and Λ have the *same* characteristic polynomial:

$$\begin{aligned} P(\lambda) &= \det(\lambda I - A) = \det\left(\lambda I - B\Lambda B^{-1}\right) = \det\left[B(\lambda I - \Lambda)B^{-1}\right] \\ &= \det B \det(\lambda I - \Lambda) \det B^{-1} = \det B \det(\lambda I - \Lambda)\frac{1}{\det B} \\ &= \det(\lambda I - \Lambda) = \det \begin{bmatrix} \lambda - \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda - \lambda_n \end{bmatrix} \\ &= \prod_{j=1}^{n} (\lambda - \lambda_j). \end{aligned}$$

It is in general difficult to determine when an $n \times n$ matrix A is diagonalizable, but there are two standard and important sufficient conditions, namely

(1) if A has n distinct eigenvalues, or
(2) if A is symmetric.

Indeed, if the characteristic polynomial of A has n *distinct* roots, then corresponding eigenvectors are linearly independent! In fact, in the simple case $n = 2$, suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to *distinct* eigenvalues λ_1 and λ_2, and *in order to derive a contradiction*, that $\mathbf{v}_2 = c\mathbf{v}_1$. Then we have

$$\lambda_2(c\mathbf{v}_1) = \lambda_2\mathbf{v}_2 = A\mathbf{v}_2 = A(c\mathbf{v}_1) = cA\mathbf{v}_1 = c\lambda_1\mathbf{v}_1,$$

which implies $c(\lambda_2 - \lambda_1)\mathbf{v}_1 = \mathbf{0}$. This contradicts $\mathbf{v}_1 \neq \mathbf{0}$ since both c and $\lambda_1 - \lambda_2$ are nonvanishing. The case when $n > 2$ can be proved in similar fashion by induction on the number ℓ of eigenvectors considered. Indeed, fix n and $1 < \ell \leq n$, and with the obvious notation, suppose that the eigenvectors $\{\mathbf{v}_1, ..., \mathbf{v}_{\ell-1}\}$ are linearly independent, and *in order to derive a contradiction*, that $\mathbf{v}_\ell = \sum_{k=1}^{\ell-1} c_k\mathbf{v}_k$. Then^{Bx}

$$\begin{aligned} \sum_{k=1}^{\ell-1} c_k\lambda_\ell\mathbf{v}_k &= \lambda_\ell\sum_{k=1}^{\ell-1} c_k\mathbf{v}_k = \lambda_\ell\mathbf{v}_\ell = A\mathbf{v}_\ell = A\left(\sum_{k=1}^{\ell-1} c_k\mathbf{v}_k\right) \\ &= \sum_{k=1}^{\ell-1} c_k A\mathbf{v}_k = \sum_{k=1}^{\ell-1} c_k\lambda_k\mathbf{v}_k, \end{aligned}$$

which implies $\sum_{k=1}^{\ell-1} c_k (\lambda_\ell - \lambda_k) \mathbf{v}_k = \mathbf{0}$. This contradicts our induction assumption that the vectors $\{\mathbf{v}_1, ..., \mathbf{v}_{\ell-1}\}$ are linearly independent, since $\lambda_\ell - \lambda_k \neq 0$ for all k, and at least one of the c_k is nonzero. Altogether we have proved the following.

LEMMA 7. *If A is an $n \times n$ matrix with n distinct eigenvalues $\{\lambda_j\}_{j=1}^n$, then A has n linearly independent eigenvectors $\{\mathbf{v}_j\}_{j=1}^n$ with $A\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for $1 \leq j \leq n$.*

The standard proof that a symmetric matrix A is diagonalizable involves elementary row and column operations, and we will not reproduce the proof here.

2.1.1. *Diagonalizable matrices.* We now return to our task of calculating the exponential of a 2×2 matrix. In the more general case of a *diagonalizable* 2×2 matrix

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix},$$

there is an invertible matrix B, i.e. $\det B \neq 0$, with the property that

$$B^{-1}AB = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

is a diagonal matrix. The diagonal entries λ_j are the *eigenvalues* of the matrix Λ, and hence also the eigenvalues of the matrix A. From above we know that

$$e^{\Lambda x} = \begin{bmatrix} e^{\lambda_1 x} & 0 \\ 0 & e^{\lambda_2 x} \end{bmatrix}.$$

Now comes the trickery. We have $A = B\Lambda B^{-1}$ and so

$$\begin{aligned} e^{Ax} &= \sum_{n=0}^{\infty} \frac{1}{n!} A^n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(B\Lambda B^{-1}\right)^n x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \overbrace{\left(B\Lambda B^{-1}\right)\left(B\Lambda B^{-1}\right)\left(B\Lambda B^{-1}\right)...\left(B\Lambda B^{-1}\right)}^{n\ times} x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} B\Lambda \left(B^{-1}B\right)\Lambda\left(B^{-1}B\right)\Lambda B...\left(B^{-1}B\right)\Lambda B^{-1}x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} B\overbrace{\Lambda\Lambda\Lambda...\Lambda}^{n\ times} B^{-1}x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} B\Lambda^n B^{-1}x^n = B\left(\sum_{n=0}^{\infty} \frac{1}{n!}\Lambda^n x^n\right) B^{-1} = Be^{\Lambda x}B^{-1}. \end{aligned} \tag{2.4}$$

Now consider the vectors $\mathbf{v}_1 = B\mathbf{e}_1$ and $\mathbf{v}_2 = B\mathbf{e}_2$. We have for each $j = 1, 2$:

$$e^{Ax}\mathbf{v}_j = Be^{\Lambda x}B^{-1}B\mathbf{e}_j = Be^{\Lambda x}\mathbf{e}_j = Be^{\lambda_j x}\mathbf{e}_j = e^{\lambda_j x}B\mathbf{e}_j = e^{\lambda_j x}\mathbf{v}_j\ .$$

Thus $e^{\lambda_j x}$ is an eigenvalue of the matrix e^{Ax} with eigenvector $\mathbf{v}_j$, and moreover, $\mathbf{y}_j = e^{\lambda_j x}\mathbf{v}_j = e^{Ax}\mathbf{v}_j$ is a *solution* to the system (2.1). Finally, from linear algebra we know that the eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ span $\mathbb{R}^2$, and so the matrix with columns $\mathbf{v}_1, \mathbf{v}_2$ is invertible. It follows that

$$\{\mathbf{y}_1, \mathbf{y}_2\} = \left\{e^{\lambda_1 x}\mathbf{v}_1, e^{\lambda_2 x}\mathbf{v}_2\right\}$$

is a *fundamental solution set* for the system (2.1). Note that the entries here are elementary functions, namely exponentials.

Finally, the above method works just the same when the matrix A is an $n \times n$ matrix with n linearly independent eigenvectors $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. But first we illustrate the method in the case $n = 2$ before giving the general theorem.

Problem 7. *Find the general solution of the system*

$$\begin{cases} y_1'(x) &= 2y_1(x) - 3y_2(x) \\ y_2'(x) &= y_1(x) - 2y_2(x) \end{cases}.$$

Solution 7. *We write the system in matrix form as*

$$\begin{aligned} \mathbf{y}'(x) &= \begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2y_1(x) - 3y_2(x) \\ y_1(x) - 2y_2(x) \end{bmatrix} \\ &= \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{y}(x) = A\mathbf{y}(x), \end{aligned}$$

and compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$*:*

$$\begin{aligned} P(\lambda) &= \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 3 \\ -1 & \lambda + 2 \end{bmatrix} \\ &= (\lambda - 2)(\lambda + 2) - (-1)(3) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1). \end{aligned}$$

So A has distinct eigenvalues ± 1. Now we compute the corresponding eigenvectors. When $\lambda = 1$ we solve

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1-2 & 3 \\ -1 & 1+2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 + 3v_2 \\ -v_1 + 3v_2 \end{bmatrix}, \end{aligned}$$

which gives $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ *is an eigenvector for* 1*. When $\lambda = -1$ we solve*

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -1-2 & 3 \\ -1 & -1+2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 + 3v_2 \\ -v_1 + v_2 \end{bmatrix}, \end{aligned}$$

which gives $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *is an eigenvector for* -1*. Thus a fundamental solution set is*

$$\{\mathbf{y}_1, \mathbf{y}_2\} = \left\{e^{\lambda_1 x}\mathbf{v}_1, e^{\lambda_2 x}\mathbf{v}_2\right\} = \left\{e^x \begin{bmatrix} 3 \\ 1 \end{bmatrix}, e^{-x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

and the general solution is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \mathbf{y}(x) = c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) = \begin{bmatrix} 3c_1e^x + c_2e^{-x} \\ c_1e^x + c_2e^{-x} \end{bmatrix}.$$

Here is the general theorem for diagonalizable matrices.

Theorem 16. *Suppose the $n \times n$ matrix A has n linearly independent eigenvectors $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$, with corresponding eigenvalues $\{\lambda_1, ..., \lambda_n\}$, where the λ_j may be real or complex, and need not be distinct. Then the system (2.1),*

$$\mathbf{y}'(x) = A\mathbf{y}(x), \qquad -\infty < x < \infty,$$

has fundamental solution set

$$\left\{e^{\lambda_1 x}\mathbf{v}_1, ..., e^{\lambda_n x}\mathbf{v}_n\right\}$$

on $(-\infty, \infty)$*, and the general solution is given by*

$$\mathbf{y}(x) = c_1 e^{\lambda_1 x}\mathbf{v}_1 + c_2 e^{\lambda_2 x}\mathbf{v}_1 ... + c_n e^{\lambda_n x}\mathbf{v}_n, \qquad -\infty < x < \infty.$$

Now we give an example to illustrate the use of variation of parameters to solve a *non*homogeneous system with a constant coefficient matrix A,

$$\mathbf{y}'(x) = A\mathbf{y}(x) + \mathbf{f}(x).$$

First we note that in this case, $\Phi(x) = e^{Ax}$ and $\Phi(t)^{-1} = e^{-At}$, and so the variation of parameters formula can be written as

$$\begin{aligned} \mathbf{y}_p &= \Phi(x)\int_{x_0}^{x}\Phi(t)^{-1}\mathbf{f}(t)\,dt \\ &= e^{Ax}\int_{x_0}^{x} e^{-At}\mathbf{f}(t)\,dt = \int_{x_0}^{x} e^{A(x-t)}\mathbf{f}(t)\,dt, \end{aligned}$$

which when $x_0 = 0$ is the convolution of the exponential matrix-valued function $\exp_A(s) = e^{As}$ with the vector-valued function $\mathbf{f}(t)$. We have used here the fact that for any matrix B, a simple computation with the power series definitions of e^B and e^{-B} shows that $e^B e^{-B} = I$, hence $\left(e^B\right)^{-1} = e^{-B}$.

Example 38. *In order to solve the* non*homogeneous system*

$$\mathbf{y}'(x) = A\mathbf{y}(x) + \mathbf{f}(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{y}(x) + \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we first compute the characteristic polynomial

$$P(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ *is* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$*, and the eigenvector corresponding to the eigenvalue* $\lambda = -1$ *is* $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$*. Thus a fundamental solution set for the homogenous system is*

$$\left\{\mathbf{y}_1(x) = e^x\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \mathbf{y}_2(x) = e^{-x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\},$$

and the complementary solution is

$$\mathbf{y}_c(x) = c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) = c_1 e^x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Finally, the variation of parameters formula for a particular solution to the nonhomogeneous system with $x_0 = 0$ *is*

$$\begin{aligned} \mathbf{y}_p(x) &= \int_0^x e^{A(x-t)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}dt = -A^{-1}e^{A(x-t)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\ |_0^x \\ &= -A^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + A^{-1}e^{Ax}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Now the second term $A^{-1}e^{Ax}\begin{pmatrix}1\\1\end{pmatrix}$ *solves the homogeneous system, so we can discard this term and use*

$$\mathbf{y}_p(x) = -A^{-1}\begin{pmatrix}1\\1\end{pmatrix} = -A\begin{pmatrix}1\\1\end{pmatrix} = -\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{pmatrix}1\\1\end{pmatrix} = -\begin{pmatrix}1\\1\end{pmatrix},$$

since $A^{-1} = A$ *for the matrix* $A = \begin{bmatrix}0&1\\1&0\end{bmatrix}$. *Thus the general solution to the nonhomogeneous system is*

$$\begin{aligned}\mathbf{y}(x) &= \mathbf{y}_c(x) + \mathbf{y}_p(x)\\ &= c_1e^x\begin{pmatrix}1\\1\end{pmatrix} + c_2e^{-x}\begin{pmatrix}1\\-1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}.\end{aligned}$$

2.1.2. *The Jordan Canonical Form.* It turns out that when some of the eigenvalues have multiplicity greater than one, there may *not* be n linearly independent eigenvectors - this corresponds to the case of repeated roots for the n^{th} order constant coefficient linear equation $L = 0$. This case can be analyzed using the Jordan Canonical Form of the matrix A (see below), and calculating the exponential of the $\ell \times \ell$ Jordan blocks

$$J_{\lambda,\ell} = \begin{bmatrix}\lambda & 1 & 0 & \cdots & 0\\ 0 & \lambda & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \lambda & 1\\ 0 & \cdots & 0 & 0 & \lambda\end{bmatrix} = \lambda I + N,$$

where the *nilpotent* matrix N satisfies $N^\ell = 0$, and

$$N^k = \begin{bmatrix}0 & \cdots & 1 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 1\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & \cdots & 0\end{bmatrix}, \qquad 1 \le k \le \ell - 1.$$

In the matrix above, the $1's$ appear on the k^{th} superdiagonal. Since λI and N commute, we then have

$$\begin{aligned}e^{J_{\lambda,\ell}x} &= e^{\lambda Ix+Nx} = e^{\lambda Ix}e^{Nx} = e^{\lambda x}I\left(\sum_{k=0}^{\ell-1}\frac{1}{k!}N^kx^k\right)\\ &= \begin{bmatrix}e^{\lambda x} & xe^{\lambda x} & \frac{x^2}{2!}e^{\lambda x} & \cdots & \frac{x^{\ell-1}}{(\ell-1)!}e^{\lambda x}\\ 0 & e^{\lambda x} & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & \frac{x^2}{2!}e^{\lambda x}\\ \vdots & \ddots & \ddots & e^{\lambda x} & xe^{\lambda x}\\ 0 & \cdots & 0 & 0 & e^{\lambda x}\end{bmatrix}.\end{aligned} \tag{2.5}$$

In the calculation above we used the fact that if two matrices A and B commute, then the binomial theorem holds for them, i.e.

$$(A+B)^n = \sum_{k+\ell=n} \frac{n!}{k!\ell!} A^k B^\ell,$$

and hence the exponent formula $e^{A+B} = e^A e^B$:

$$\begin{aligned} e^{A+B} &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k+\ell=n} \frac{n!}{k!\ell!} A^k B^\ell \\ &= \sum_{k,\ell=0}^{\infty} \frac{1}{k!\ell!} A^k B^\ell \\ &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k \right) \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} B^\ell \right) = e^A e^B. \end{aligned}$$

Recall that every $n \times n$ matrix A has a Jordan Canonical Form J that consists of Jordan blocks $J_{\lambda,\ell}$ along the main diagonal, where λ is an eigenvalue of A and ℓ is the size of the block. More precisely, there is an invertible $n \times n$ matrix B such that

$$B^{-1}AB = J = \begin{bmatrix} \left[J_{\lambda_1,\ell_{1,1}}\right] & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \ddots & \left[J_{\lambda_1,\ell_{1,q_1}}\right] & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \left[J_{\lambda_k,\ell_{k,1}}\right] & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \left[J_{\lambda_k,\ell_{k,q_k}}\right] \end{bmatrix},$$

where $\lambda_1, ..., \lambda_k$ are the eigenvalues of A, and $\ell_{j,1} + ... + \ell_{j,q_j} = m_j$ is the multiplicity of the eigenvalue λ_j for $1 \le j \le k$. Thus by the calculation in (2.4) we have the formula

$$e^{Ax} = e^{BJB^{-1}x} = Be^{Jx}B^{-1} \tag{2.6}$$

$$= B \begin{bmatrix} \left[e^{J_{\lambda_1,\ell_{1,1}}x}\right] & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \ddots & \left[e^{J_{\lambda_1,\ell_{1,q_1}}x}\right] & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \left[e^{J_{\lambda_k,\ell_{k,1}}x}\right] & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \left[e^{J_{\lambda_k,\ell_{k,q_k}}x}\right] \end{bmatrix} B^{-1},$$

where the exponentials of the Jordan blocks are given by (2.5).

CONCLUSION 4. *Thus a fundamental solution set for the system* $\mathbf{y}' = A\mathbf{y}$ *is given by the columns of the matrix* $Be^{Jx}B^{-1}$*, and we see that the entries in the* *columns are linear combinations of polynomials in x times exponentials $e^{\lambda x}$ where λ runs through eigenvalues of A.*

Remark 8. *In the case $n = 2$, an arbitrary matrix A has either exactly one eigenvector, or exactly two eigenvectors. In the case A has exactly one eigenvector, with eigenvalue λ, the Jordan Canonical Form consists of a single 2×2 block $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. If $B^{-1}AB = J$, then*

$$e^{Ax} = e^{BJB^{-1}x} = Be^{Jx}B^{-1} = B\begin{bmatrix} e^{\lambda x} & xe^{\lambda x} \\ 0 & e^{\lambda x} \end{bmatrix}B^{-1}. \tag{2.7}$$

Here is a simple example that illustrates the use this result.

Example 39. *In order to solve the system*

$$\mathbf{y}' = A\mathbf{y} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\mathbf{y},$$

we compute the characteristic polynomial

$$P(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 3 \end{bmatrix} = (\lambda - 2)^2,$$

and see that $\lambda = 2$ is the only eigenvalue of A, and has multiplicity 2. Now for $\lambda = 2$ we have

$$2I - A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

and the only eigenvector (up to multiples) is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus one solution of the system is

$$\mathbf{y}_1(x) = e^{2x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The Jordan Canonical Form for A consists of a single Jordan block $J_{2,2}$,

$$B^{-1}AB = J = J_{2,2} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

and from formula (2.6) or (2.7) for the exponential e^{Ax}, we see that

$$e^{Ax} = B\begin{bmatrix} e^{2x} & xe^{2x} \\ 0 & e^{2x} \end{bmatrix}B^{-1}.$$

Thus there is a second solution of the form

$$\mathbf{y}_2(x) = xe^{2x}\mathbf{v} + e^{2x}\mathbf{w},$$

for certain vectors $\mathbf{v}$ and $\mathbf{w}$ that can be evalutated by plugging into the system:

$$\left(e^{2x} + 2xe^{2x}\right)\mathbf{v} + 2e^{2x}\mathbf{w} = \mathbf{y}_2'(x) = A\mathbf{y}_2 = xe^{2x}A\mathbf{v} + e^{2x}A\mathbf{w}.$$

Equating coefficients of the independent functions e^{2x} and xe^{2x} we get

$$\begin{aligned} (2I - A)\mathbf{v} &= \mathbf{0}, \\ (2I - A)\mathbf{w} &= -\mathbf{v}. \end{aligned}$$

Now $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and* $\mathbf{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ *satisfy these equations, and so a second independent solution is given by*

$$\mathbf{y}_2(x) = xe^{2x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2x}\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The general solution is then given by

$$\begin{aligned} \mathbf{y}(x) &= c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) \\ &= \left(c_1 e^{2x} + c_2 x e^{2x}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2x}\begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{aligned}$$