On the Boltzmann equation: global solutions in one spatial dimension

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Summary

The Boltzmann equation for $f(x, v, t) \geq 0$ with $x \in \mathbb{R}^1 / \mathbb{Z}^1$, $v \in \mathbb{R}^3$

The macroscopic density

$$\rho(x, t) = \int_{v \in \mathbb{R}^3} f(x, v, t) \, dv$$

The entropy relative to a Maxwellian $M(v) = m \left( \frac{a}{\pi} \right)^{3/2} e^{-a|v-u|^2}$ is

$$H(f|M) = \int_{x \in \mathbb{T}^1} \int_{v \in \mathbb{R}^3} \left( f \log\left( \frac{f}{M} \right) - f + M \right) \, dv \, dx$$

Main result: There exists a constant $C_0$ such that if $\rho_0(x) \in L_x^\infty$ and

for some $M_0$, $H(f_0|M_0) \leq C_0$ then global strong solutions exist.
Outline

The Boltzmann equation

Global existence results

Uniqueness

Properties of propagation

Main ideas of the proof

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The classical Boltzmann equation

\[ \partial_t f + v \cdot \partial_x f = Q(f, f), \quad f(x, v, 0) = f_0(x, v) \]  \hspace{1cm} (1)

Phase space coordinates \((x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v\).

Phase space distribution function \(f(x, v, t)\).

In case \(Q = 0\) the solution is

\[ f(x, v, t) = f_0(x - tv, v) = \Phi_t(f_0)(x, v) \]

Free streaming flow of the equation (1) linearized about \(f = 0\)

\(Q(f, f)\) is the collision operator
Boltzmann equation

The classical Boltzmann equation

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- The classical **Boltzmann equation**

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**Free streaming flow** of the equation (1) linearized about \(f = 0\)

- \(Q(f, f)\) is the **collision operator**
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Main ideas of the proof

portrait of Ludwig Boltzmann (1844 - 1906)

29/09/05
http://www.sil.si.edu/digitalcollection-identity/thumbnails/TNSIL14-B5-06.jpg

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Global solutions of the Boltzmann equation
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portrait of J. C. Maxwell (1831 - 1879)

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Global solutions of the Boltzmann equation
Collision operator

- The collision operator of Maxwell and Boltzmann

\[ Q(f, f)(x, v) = \int_{\mathbb{R}^3_{v_0}} \int_{S^2_{\sigma}} (f(x, v')f(x, v_*) - f(x, v)f(x, v_*)) \times K dS_{\sigma} dv_* \]

\[ = Q^+(f, f)(x, v) - Q^-(f, f)(x, v) \]  \hspace{1cm} (2)

- Velocities before \( v', v_\ast \) and after \( v, v_\ast \) a binary collision satisfy

\[ v + v_\ast = v' + v_\ast \quad \text{and} \quad (v' - v_\ast)/|v - v_\ast| = \sigma \in S^2 \]

- The collision kernel \( K = K(|v - v_\ast|, \frac{(v - v_\ast)}{|v - v_\ast|} \cdot \sigma) \)
Collision operator

The collision operator of Maxwell and Boltzmann

\[ Q(f,f)(x,v) = \int_{\mathbb{R}^3_v} \int_{\mathbb{S}^2_\sigma} (f(x,v')f(x,v_*) - f(x,v)f(x,v_*)) \times K dS_\sigma dv_* \]

\[ = Q^+(f,f)(x,v) - Q^-(f,f)(x,v) \quad (2) \]

Velocities before \(v', v_*\) and after \(v, v_*\) a binary collision satisfy

\[ v + v_* = v' + v'_* \quad \text{and} \quad (v' - v'_*)/(v - v_*) = \sigma \in \mathbb{S}^2 \]

The collision kernel

\[ K = K(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma) \]
Collision operator

- The collision operator of Maxwell and Boltzmann

\[
Q(f,f) (x,v) = \int_{\mathbb{R}^3} \int_{S^2} (f(x,v')f(x,v_*) - f(x,v)f(x,v_*)) \times K \, dS_\sigma \, dv_*
\]

\[
= Q^+(f,f)(x,v) - Q^-(f,f)(x,v)
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(2)

- Velocities before \(v', v_*\) and after \(v, v_*\) a binary collision satisfy

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\]

- The collision kernel \(K = K(|v - v_*|, \frac{(v-v_*)}{|v-v_*|} \cdot \sigma)\)
Macroscopic quantities

- macroscopic mass density

\[ \rho(x, t) = \int_{v \in \mathbb{R}^3} f(x, v, t) \, dv \]

- macroscopic momentum density

\[ \rho u(x, t) = \int_{v \in \mathbb{R}^3} v f(x, v, t) \, dv \]

- macroscopic energy density

\[ \rho e(x, t) = \int_{v \in \mathbb{R}^3} |v|^2 f(x, v, t) \, dv \]
Macroscopic conservation laws

- **Conservation of mass:**
  \[ \partial_t \rho(x, t) + \nabla_x \cdot F_\rho(x, t) = 0 \]
  with density flux \( F_\rho = \int_{v \in \mathbb{R}^3} vf(x, v, t) \, dv \)

- **Conservation of momentum:**
  \[ \partial_t \rho u(x, t) + \nabla_x \cdot F_{\rho u}(x, t) = 0 \]
  with momentum flux \( F_{\rho u} = \int_{v \in \mathbb{R}^3} v \otimes vf(x, v, t) \, dv \)

- **Conservation of energy:**
  \[ \partial_t \rho e(x, t) + \nabla_x \cdot F_{\rho e}(x, t) = 0 \]
  with energy flux \( F_{\rho e} = \int_{v \in \mathbb{R}^3} |v|^2 vf(x, v, t) \, dv \)

NB The closure problem
One dimensional geometry

We take $x \in \mathbb{R}^1$ and $v \in \mathbb{R}^3$, which is known as the slab geometry. It is the setting of a narrow shock tube, or a situation with spanwise constant macroscopic quantities.

Furthermore, we ask for periodic spatial boundary conditions; that is $x \in \mathbb{R}^1$ and $v \in \mathbb{R}^3$.

**NB:** Periodic boundary conditions, or perturbations of a state of thermodynamic equilibrium (a constant Maxwellian background) are more difficult problems than setting $f \in L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ in the background of a vacuum.
Conserved quantities

- **Total Mass**

\[ M(f) = \int_{\mathbb{T}^1} \rho(x, t) \, dx = \int_{\mathbb{T}^1} \int_{v \in \mathbb{R}^3} f(x, v, t) \, dv \, dx \]

- **Total Momentum**

\[ I(f) = \int_{\mathbb{T}^1} \rho u(x, t) \, dx = \int_{\mathbb{T}^1} \int_{v \in \mathbb{R}^3} v f(x, v, t) \, dv \, dx \]

- **Total Energy**

\[ E(f) = \int_{\mathbb{T}^1} \rho e(x, t) \, dx = \int_{\mathbb{T}^1} \int_{v \in \mathbb{R}^3} |v|^2 f(x, v, t) \, dv \, dx \]

These quantities are constants of motion for the Boltzmann equation.
Relative entropy

Define the relative entropy of a phase space distribution function \( f(x, v) \) with respect to a constant Maxwellian distribution by

\[
H(f|M) = \int_{T^1} \int_{v \in \mathbb{R}^3} \left( f \log \left( \frac{f}{M} \right) - f + M \right) dv dx
\]

Proposition (1)

Positivity properties of the relative entropy

1. \( H(f|M) \geq 0 \)

2. Let \( m := \int_{v \in \mathbb{R}^3} M dv \). The Csiszár–Kullback–Pinsker inequality is equivalent to

\[
\int_{T^1} \int_{v \in \mathbb{R}^3} |f(x, v) - M(v)| dv dx \leq \sqrt{4mH(f|M) + H(f|M)}
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Relative entropy

Define the relative entropy of a phase space distribution function $f(x, v)$ with respect to a constant Maxwellian distribution by

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The $H$-theorem

Proposition (2)

The relative entropy $H(f|M)$ decreases along the flow of the Boltzmann equation

$$\partial_t H(f|M) \leq 0$$

Remarks: The following are equivalent:

1. $f \in L \log(L)$ and $\int_x \int_v (1 + |v|^2) |f| dvdx < +\infty$
2. $f \in L \log \left( \frac{|f|}{M} \right)$ for some Maxwellian $M(v)$
The $\|f\|$ norm

**Definition (3)**

For functions $f(x, v)$ on phase space define the norm

$$\|f\| := \sup_{x \in \mathbb{T}^1} \int_{\mathbb{R}^3_v} |f|(x - pv, v) \, dv$$  \hspace{1cm} (3)

Denote by $X$ the space of functions for which this is finite.

The function space $X \subseteq L_x^\infty(L_v^1)$.

**NB:** The streaming flow $\Phi_t(f)(x, v) = f(x - tv, v)$ is not continuous on $L_x^\infty(L_v^1)$ but it is so on $X$. 

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Global solutions of the Boltzmann equation
The $\|f\|$ norm

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For functions $f(x, v)$ on phase space define the norm

$$\|f\| := \sup_{x \in T^1} \int_{\mathbb{R}^3_v} |f(x - pv, v)| dv$$  \hspace{1cm} (3)

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The function space $X \subseteq L^\infty_x(L^1_v)$.

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First global result

Theorem (4)

Assume that the collision kernel $K$ satisfies hypothesis $(H1)$. Then there is a constant $K_0$ such that if initial data $f_0$ for the Boltzmann equation (1) satisfies

$$
\|f_0\| < +\infty \quad \text{and} \quad H(f_0|M_0) \leq \frac{1}{4\pi K_0}
$$

for some Maxwellian $M_0$, then the solution $f(\cdot, t) \in X$ for all $t \in \mathbb{R}^+$. More specifically the theorem asserts that there is an estimate on the $L_x^\infty$ norm of the density $\rho(x, t)$. When strict inequality holds in (4)

$$
\exists \beta \quad \text{such that} \quad \|\rho(x, t)\|_{L_x^\infty} \leq \|f(x, v, t)\| \leq C_0 \exp(\sqrt{t/\beta})
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\exists \beta \quad \text{such that} \quad \| \rho(x, t) \|_{L_x^\infty} \leq \| f(x, v, t) \| \leq C_0 \exp\left(\sqrt{t/\beta}\right)
$$
Hypotheses on the collision kernel

The result depends upon a newly described smoothing property of the collision kernel $K$.

$$K(r, \xi) = K\left(|v - v_*|, \frac{(v - v_*) \cdot \sigma}{|v - v_*|}\right)$$

Hypothesis (H1)

*Large relative velocity interactions are soft, while small relative velocity interactions are hard:*

$$(H1) \quad 0 \leq K(r, \xi) \leq \frac{K_0 w}{1 + w \log^{1+\varepsilon}(w + 1)}$$

NB This hypothesis is related to one appearing in Cercignani’s work.
Compare with Boltzmann’s collision kernel

Power law potential interactions between molecules for $1 < p \leq \infty$ are in the form

$$V(|q_1 - q_2|) = \frac{\gamma}{|q_1 - q_2|^p}$$

In this case, the classical Boltzmann collision kernel has the form

$$K(r, \xi) = b(\xi)|v - v_*|^{\beta}$$

where $-\infty < \beta = \beta(p) \leq 1$.

For hard spheres $\beta = 1$, For the Maxwell molecule case $p = 5$ it turns out that $\beta = 0$.

**NB:** For all but the hard spheres case, $b(\xi)$ diverges nonintegrably as $\xi \to \pm 1$. **Grad cutoffs** appear here, where they truncate grazing collisions.
A stronger hypothesis on the collision kernel gives a better result

**Hypothesis (H2)**

*Small relative velocity interactions are absent.*

(i) \( K(r, \xi) = 0 \) for \( r < R_1 \)

(ii) \( K(r, \xi) \geq \beta(\xi) \sup_{\xi \in (-1,1)} (K(r, \xi)) \) for \( R_1 < r \),

*where \( \beta(\xi) \) is positive on a set of positive measure.*

**Theorem (5)**

*Assume that hypotheses (H1) and (H2) hold, and that*

\[ \|f_0\| < +\infty \quad \text{and} \quad M(f_0) + E(f_0) < +\infty \quad (5) \]

*then for all \( t \in \mathbb{R}^+ \) solutions to (1) exist and have \( L^\infty_x \) macroscopic density \( \rho(x, t) \).

**NB** These solutions could have infinite entropy
A stronger hypothesis on the collision kernel gives a better result

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**NB** These solutions could have infinite entropy
Uniqueness in $X$

- Solutions $f(x, v, t)$ of the Boltzmann equation (1) such that
  \[ \rho(x, t) = \int_{\mathbb{R}^3_v} f(x, v, t) \, dv \in C([0, T] : L_x^\infty) \]

  are known as **strong** solutions.

- Solutions in the class $X$ are unique.

- However the more general property of **weak/strong** uniqueness holds.
Uniqueness in $X$

- Solutions $f(x, v, t)$ of the Boltzmann equation (1) such that

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Dissipative solutions

Definition (P.-L. Lions (1994))

A nonnegative $f(x, v, t) \in C([0, T]; L^1(\mathbb{T}^1 \times \mathbb{R}^3))$ is a dissipative solution of the Boltzmann equation (1) if for all

$$g(x, v, t) \in L^\infty([0, T]; L^\infty_x(\mathbb{T}^1; L^1_v(\mathbb{R}^3)))$$

satisfying

$$\partial_t g + v \cdot \nabla_x g - Q(g, g) = \mathcal{E}(g),$$

in the sense of distributions, then $f$ obeys the inequality

$$\partial_t \int |f - g| \, dv + \nabla_x \cdot \int v|f - g| \, dv \leq \int Q(g, f - g) \text{sgn}(f - g) \, dv - \int \mathcal{E}(g) \text{sgn}(f - g) \, dv$$
Weak/strong uniqueness

Theorem (6)

Given initial data satisfying (4), that is

\[ \|f_0\| < +\infty \quad \text{and} \quad H(f_0|M_0) \leq \frac{1}{4\pi K_0} \tag{6} \]

any other dissipative solution starting with the same initial data \( f_0 \) must coincide with the strong solution for all \( t \in \mathbb{R}^+ \)

In the case where hypothesis \((H2)\) also holds, for uniqueness we only require

\[ \|f_0\| \quad \text{and} \quad M(f_0) + E(f_0) < +\infty \]
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Moments and derivatives

This question has to do with the evolution of smoothness and moment properties of the phase space distribution function $f(x, v, t)$, once the basic property of being a strong solution is established.

- **Moments**
  \[ M_\kappa(f) = \sup_{x \in \mathbb{T}^1} \int_{\mathbb{R}^3_\nu} \nu^\kappa f(x, v, t) \, dv \]

- **Derivatives in $v$**
  \[ Q_\lambda(f) = \sup_{x \in \mathbb{T}^1} \int_{\mathbb{R}^3_\nu} |\partial^\lambda_v f(x, v, t)| \, dv \]

- **Derivatives in $x$**
  \[ P_\mu(f) = \sup_{x \in \mathbb{T}^1} \int_{\mathbb{R}^3_\nu} |\partial^\mu_x f(x, v, t)| \, dv \]
Propagation of moments and derivatives

Theorem (7)

Given initial data $f_0(x, v)$ satisfying (4) (or in the case where hypothesis (H2) also holds, (5)). If in addition for integers $k, \ell, m$ the data satisfies

$$
\sum_{|\kappa| \leq k, |\lambda| \leq \ell, |\mu| \leq m} \sup_{x \in \mathbb{T}^1} \int_{v \in \mathbb{R}^3} |v^\kappa| |\partial_v^\lambda \partial_x^\mu f_0(x, v)| \, dv \, dx < +\infty
$$

then for all $t > 0$

$$
\sum_{|\kappa| \leq k, |\lambda| \leq \ell, |\mu| \leq m} \sup_{x \in \mathbb{T}^1} \int_{v \in \mathbb{R}^3} |v^\kappa| |\partial_v^\lambda \partial_x^\mu f(x, v, t)| \, dv \, dx \leq \varphi_{k\ell m}(t) < +\infty
$$

The growth rate is bounded by $c_{k\ell m} \exp(\exp(\sqrt{t/\beta}))$. 
Propagation of moments and derivatives

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Given initial data $f_0(x, v)$ satisfying (4) (or in the case where hypothesis (H2) also holds, (5)). If in addition for integers $k, \ell, m$ the data satisfies

$$\sum_{|\kappa| \leq k, |\lambda| \leq \ell, |\mu| \leq m} \sup_{x \in \mathbb{T}^1} \int_{v \in \mathbb{R}^3} |v^\kappa||\partial^\lambda_v \partial^\mu_x f_0(x, v)| \, dv \, dx < +\infty$$

then for all $t > 0$

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The growth rate is bounded by $c_{k\ell m} \exp(\exp(\sqrt{t/\beta}))$. 

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Global solutions of the Boltzmann equation
Main ideas of the proof

- Using Duhamel’s principle and the streaming flow, rewrite the Boltzmann equation as an integral equation

\[ f(x, v, t) = f_0(x - tv, v) + \int_0^t Q(x - (t - s)v, v, s) \, ds \]

- Solutions satisfy the maximum principle:
  \[ f_0(x, v) \geq 0 \quad \text{implies} \quad f(x, v, t) \geq 0 \quad \text{for} \quad t > 0 \]

- Drop the \( Q^- \) term for the inequality

\[ 0 \leq f(x, v, t) \leq f_0(x - tv, v) + \int_0^t Q^+(x - (t - s)v, v, s) \, ds \]

- Integrate in \( v \in \mathbb{R}^3 \)

\[ 0 \leq \rho(x, t) \leq \int_v f_0(x - tv, v) \, dv + \int_0^t \int_v Q^+(x - (t - s)v, v, s) \, dvds \]  
(7)

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Global solutions of the Boltzmann equation
The integrand in (7) involves a spherical integral through the part of the collision operator $Q^+$

\[
\int_v Q^+ (x - (t - s)v, v, s) \, dv
\]

\[
= \int_v \int_{v_*} \int_{S^2_{\sigma}} f(x - (t - s)v, v') f(x - (t - s)v, v'_*) \times K(|v - v_*|, \frac{(v - v_*)}{|v - v_*|} \cdot \sigma) \, dS_{\sigma} \, dv_* \, dv
\]

\[
= \int_v \int_{v_*} \int_{S^2_{\sigma}} f(x - (t - s)v', v) f(x - (t - s)v', v_*) \times K(|v - v_*|, \frac{(v - v_*)}{|v - v_*|} \cdot \sigma) \, dS_{\sigma} \, dv_* \, dv
\]

where $v' = \frac{1}{2}((v + v_*) + |v - v_*| \sigma)$
A smoothing property

In one dimensional geometries, the integral over $\sigma \in S^2$ converts to a spatial integral over the interval $[v_{\text{min}}, v_{\text{max}}]$ where

$$v_{\text{min}} = \frac{1}{2}((v + v_*) \cdot e_1 - |v - v_*|)$$
$$v_{\text{max}} = \frac{1}{2}((v + v_*) \cdot e_1 + |v - v_*|)$$

Changing variables in the integral (assume for simplicity that $K(r, \xi) = 0$ for $r > R$)

$$\int_v Q^+(x - (t - s)v, v, s) \, dv \leq \frac{1}{t - s} \int_{-R(t-s)}^{R(t-s)} 4\pi K_0 \rho^2(x + z, s) \, dz$$
Two estimates of the integrand

- The first estimate of the integrand is simply through $\|\rho\|_{L^\infty}$

$$\frac{1}{t - s} \int_{-R(t-s)}^{R(t-s)} 4\pi K_0 \rho^2(x + z, s) \, dz \leq 8\pi K_0 R \|\rho\|_{L^\infty}^2$$

- The second estimate uses the relative entropy $H(f|M)$

$$\frac{1}{t - s} \int_{-R(t-s)}^{R(t-s)} 4\pi K_0 \rho^2(x + z, s) \, dz \leq \frac{4\pi K_0 H(f|M)}{t - s} \frac{\|\rho\|_{L^\infty}}{\log(\|\rho\|_{L^\infty})}$$

Each one individually gives rise to an estimate which doesn’t forbid blowup in finite time.
The integral inequality

Estimate the integrand by optimizing the two estimates
Denote $\|f(x, v, t)\| = \varphi(t)$ and $\alpha = C_0K_0H(f_0|M)$

$$
\varphi(t) \leq \varphi(0) + \int_0^t \min \left\{ c_1\varphi^2(s), \left( \frac{\alpha}{t-s} + c_2 \right) \frac{\varphi(s)}{\log(\varphi(s))} \right\} ds \quad (8)
$$

Theorem (8)

*Global solutions of (8) depend only upon $\alpha \leq 1$.*

The constant $c_1$ depends upon $\varepsilon, K_0$ and $H(f_0|M)$, while $c_2$ depends upon the initial data.
The Bony functional

- The proof of the second theorem under hypothesis (H2) is based on the Bony functional

\[ b(t) = \int \int \int \int f(x, v, t) f(x, v_*, t) \]
\[ \times K(|v - v_*|, \sigma \cdot (v - v_*)/|v - v_*|)|v - v_*|^2 dS_\sigma dv dv_* dx \]

which has the property that \( \int_0^\infty b(t) \, dt < +\infty \)

- Using similar ideas in the case where (H2) holds, the integral inequality is

\[ \varphi(t) \leq \varphi(0) + \int_0^t \min \left\{ c_1 \varphi^2(s), \left( \frac{1}{t - s} + 1 \right) c_3 b(s) \right\} ds \quad (9) \]
global existence

Proposition (9)

Given $b(t)$, the maximal solution $\varphi(t)$ of the integral inequality (9) is a locally bounded function of $t$.

Furthermore the quantity $\|f(\cdot, t)\| \leq \varphi(t)$. This implies global existence of a strong solution $f(x, v, t)$

However no rate of growth is available from this method, and indeed there may not be any quantitative rate.
Future directions

- Theorem (5) is restricted to one dimensional geometries. It is an open question whether Theorem (4) is so constrained.
- Can we relax the conditions \((H1)\) on the collision kernel, possibly by using energy conservation.
- Does a similar theorem hold for nonelastic collisions (P. Degond). One of the difficult tendencies is for particles to coagulate (this is avoided precisely through hypothesis \((H2)\) in our second category of results).
- Use Boltzmann-like kinetic equations to study homogeneous forms of dispersive nonlinear Hamiltonian partial differential equations