Towards a New Proof of Anderson Localization

Robert Brandenberger[1] and Walter Craig[2]

1) Department of Physics,
McGill University, 3600 Rue Université,
Montréal H3A 2T8, Québec, Canada
E-mail: rhb@physics.mcgill.ca and
2) Department of Mathematics and Statistics, McMaster University
Hamilton, Ontario, L8S 4K1, Canada
E-mail: craig@math.mcmaster.ca

The wave function of a non-relativistic particle in a periodic potential admits oscillatory solutions, the Bloch waves. In the presence of a random noise contribution to the potential the wave function is localized. We outline a new proof of this Anderson localization phenomenon in one spatial dimension, extending the classical result to the case of a periodic background potential. The proof makes use of techniques previously developed to study the effects of noise on reheating in inflationary cosmology, employing methods of random matrix theory.

I. INTRODUCTION

A classic problem in non-relativistic quantum mechanics is the propagation of a particle, e.g. an electron, in a periodic potential, set up e.g. by a lattice of ions. It is known that the time-independent Schrödinger equation for this problem admits modulated periodic solutions, the so-called Bloch waves [1]. In the real world, however, we expect that the potential will not be perfectly periodic due to the presence of various types of disorder, e.g. thermal noise. As was first shown by Anderson [2], if the noise is modeled as a random contribution to the potential, then for sufficiently large disorder the wave functions become localized. In one spatial dimension, a stronger result holds: it can be shown [3] that for any amount of disorder, the wave functions become exponentially localized, i.e. instead of being oscillatory, the solutions now are exponentially decaying in space about the location of the center of the wave packet. This result was extended to two spatial dimensions in [4]. Anderson localization is reviewed in [5–7]. A mathematically rigorous proof of one dimensional Anderson localization for constant background potential is given in [8], and there is an extensive mathematical literature on the subject.

As is well known, in one spatial dimension the time-independent Schrödinger equation can be mapped into a classical time-dependent wave equation by a simple variable transformation in which the wave function Ψ becomes a classical field χ and the spatial variable x is transformed to time t. After this mapping, the Schrödinger equation for the wave function Ψ in a potential which is the sum of a periodic term $V_p(x)$ and a random noise term $V_R(x)$ becomes the relativistic wave equation (in momentum space) for a scalar field with a mass which contains one contribution to the mass which is periodically varying in time, and a second contribution which is random in time.

The inflationary scenario [9] (see also [10] for a recent review emphasizing both successes and problems of inflation), the current paradigm of early universe cosmology, is based on the dynamics of classical scalar fields coupled as matter source to Einstein's theory of General Relativity. According to the inflationary universe scenario, there is a period in the very early universe in which space expands almost exponentially. This is obtained by making use of a scalar field φ whose energy functional is dominated by a potential energy density contribution which is almost constant in time. Regular matter can be modeled as a second scalar field χ.

During the period of inflation, an exponentially increasing fraction of the energy density of the universe is stored in the field φ. Hence, to make contact with what is observed today, a phase at the end of the period of inflation when the energy transfers from φ to regular matter is crucial. This phase is called the "reheating stage", and works in the following way. The period of inflation terminates once φ begins to oscillate about the minimum of its potential. Due to a resonant coupling between φ and the matter field χ, energy can be transferred from φ to χ. The theory of reheating is obtained by studying the evolution of χ in the presence of an oscillating φ field.

The equation of motion for χ is the Klein-Gordon equation with a correction term due to the coupling with φ. Treating χ at the linearized level, each Fourier mode of χ evolved independently. Neglecting the effects of the expansion of the universe [27], and for the specific coupling between the two fields which we specify below, the equation of motion for such a Fourier mode of χ is that of a harmonic oscillator whose mass has two time-dependent contributions, one periodic in time (due to the oscillatory dynamics of φ during the reheating period, the other being an aperiodic random noise term (due e.g. to quantum fluctuations in φ).

In the absence of noise, the equation of motion is the Mathieu equation and falls into the class of equations first studied by Hill, Floquet and Legendre [11]. As a function of the value of the bare mass (the term in the mass independent of time), there are stability bands (for which the solutions are oscillatory) and instability bands (for which the solutions are characterized by overall exponential growth or decay. The coefficient describing the

exponential growth is called the "Lyapunov exponent". Applied to the case of cosmology, then in the absence of noise there are bands of Fourier modes for which the mass lies in the instability band and for which there is resonant increase in the amplitude of χ, which in terms of physics corresponds to resonant production of particles [12].

Not too long ago [13], the effects of a particular type of random noise on the resonant production of particles during inflationary reheating was studied. It was shown that the Lyapunov exponent $\mu_k(q)$ which describes the exponential growth of each Fourier mode k of the scalar field in the presence of the noise q is strictly larger than the corresponding Lyapunov exponent $\mu_k(0)$ in the absence of noise [28]. In particular, this implies that in the presence of noise, every Fourier mode grows exponentially. The stability bands which are present in the absence of noise disappear.

In this Letter, we review the above analysis, translate the results into the language of the time-independent Schrödinger equation, and in this way immediately obtain a new proof of Anderson localization which extends to the case of a periodic background potential.

II. REVIEW OF RESULTS ON THE EFFECTS OF NOISE ON REHEATING IN INFLATIONARY COSMOLOGY

We begin by reviewing the results of [13]. We consider the simplest model which describes both the inflationary phase of the very early Universe and the period of "reheating" which terminates the phase of inflation and during which the energy of the Universe is transferred to regular matter. This model contains two scalar matter fields, φ and χ. The first, φ, has a large potential energy which is changing only very slowly in time, dominates the total energy density of the Universe and thus gives rise to inflation [9]. The second scalar field, χ, represents regular matter. It is in its vacuum state initially and gets excited by a coupling with φ during the final stages of inflation when $\varphi(t)$ oscillates about the minimum of its potential [29]. It is of great interest in cosmology to study the dynamics of matter production at the end of inflation.

We will assume that the interaction Lagrangian density takes the form [30]

$$\mathcal{L}_{\text{int}} = \frac{1}{2} g \varphi \chi^2 \,, \tag{1}$$

and that the Lagrangian for χ alone is that of a free scalar field with mass m_χ. We will also consider φ to be spatially homogeneous. In the absence of quantum fluctuations, this is a reasonable assumption since during the period of inflation, spatial fluctuations are red-shifted exponentially. In this case, the equation of motion for χ can be solved independently for each Fourier mode. In the absence of expansion of the Universe, the resulting equation is [31]

$$\ddot{\chi}_k + \left[\omega_k^2 + g\varphi(t) \right] \chi_k = 0 \,, \tag{2}$$

where $\omega_k^2 = k^2 + m_\chi^2$.

At the end of the period of inflation, the scalar field φ is oscillating about its ground state value, which we without loss of generality can take to be $\varphi = 0$. The frequency ω of the oscillation is set by the mass of φ. In this case, the equation (2) has the form of a Mathieu equation. As is well known (see e.g. [11, 20, 21]), there are bands of values of ω_k for which χ_k grows exponentially. The exponential growth is governed by a Lyapunov exponent $\mu(k)$, which can be extracted from the time evolution of $\chi_k(t)$ as follows:

$$\mu_k = \lim_{t\to\infty} \frac{1}{t} log|\chi_k(t)| \,. \tag{3}$$

The time scale of exponential growth is in many concrete inflationary models much shorter than the typical expansion time of the Universe, thus justifying the neglect of the expansion. Thus [12], the parametric instability leads to a very efficient energy transfer [32]

However, in cosmology we expect fluctuations of thermal or quantum nature to be super-imposed on the homogeneous oscillation of φ. In fact, we believe that quantum vacuum fluctuations of φ in inflationary cosmology are the seeds for the inhomogeneities in the galaxy distribution and the anisotropies in the temperature of the cosmic microwave background observed today. Thus, it becomes important to study the sensitivity of the parametric resonance instability of (2) in the presence of an oscillating $\varphi(t)$ to the presence of random noise in $\varphi(t)$. In [13], this problem was studied for homogeneous, aperiodic noise, modeled as the addition of a stochastic contribution to the time dependent mass. More precisely, we studied the equation

$$\ddot{\chi}_k + \left[\omega_k^2 + p(\omega t) + q(t) \right] \chi_k = 0 \,, \tag{4}$$

where p is a periodic function with period 2π and $q(t)$ is an aperiodic, random noise contribution [33]

To derive rigorous inequalities on the magnitudes of the Lyapunov exponent with and without noise, it is necessary to make certain assumptions on the noise $q(t)$. We consider noise drawn from some sample space $\Omega = C(\mathcal{R})$ with a translationally invariant probability measure $dP(\kappa)$, where κ labels an element in Ω, and assume:

- The noise is ergodic, i.e. the ensemble average of a function of the noise equals the time average for almost all realizations of the noise.

- The noise is uncorrelated in time on scales larger than T, the period of the oscillatory contribution to the mass in (4).

- Restricting the noise to the time interval $0 \le t < T$, the samples $q(t; \kappa) : 0 \le t < T$ within the support of the probability measure fill a neighborhood of the origin in Ω.

Given these assumptions, we were able to use a theorem of random matrix theory (applied to the transfer matrix corresponding to the differential equation (4)) to prove that for almost all realizations of the noise

$$\mu(q) > \mu(0) \qquad (5)$$

for all values of the momentum k. In particular, this implies that the stability bands of the Mathieu equation disappear once noise of the type considered here is added to the system. The noise in fact strengthens the instability. Note that the result (5) was proven non-perturbatively in [13] for values of k in the stability bands, the case of interest to us here. We conjecture that the result is true in general, and have been able to show this at least perturbatively [14].

III. APPLICATION TO ANDERSON LOCALIZATION

Let us start from Eq. (4) and transform variables. We replace the time coordinate t by a spatial coordinate x (space being infinite, i.e. $\mathcal{R}$), and substitute the field variable χ by the variable ψ representing a wave function. With these substitutions, Eq. (4) becomes the time-independent Schrödinger equation for the wave function ψ in one spatial dimension for a system with a potential which is the superposition of a periodic piece $V_p(x)$ (coming from $p(\omega t)$ in (4)) and a random noise piece $V_R(x)$ coming from $q(t)$ in (4):

$$H\psi = E\psi \qquad (6)$$

with

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V_p(x) + V_R(x)\,. \qquad (7)$$

The energy eigenvalue E is given by the momentum k of (4). We use units in which $\hbar = 1$.

In the absence of noise, the spectrum of the Schrödinger equation (6) consists of bands of continuous spectrum, for which bounded quasi-periodic (modulated periodic) solutions exist, alternating with instability gaps (resonance intervals) in which the solution behavior is exponential. In both regions the solutions, the so-called Bloch waves [1], exhibit Floquet behavior, namely they are of the form $\psi(x) = \exp(\pm(\mu_E + i\alpha_E)x)p(x)$, where $p(x)$ is periodic of the same period as the potential V_p. The Floquet exponent $m_E = (\mu_E + i\alpha_E)$ has real part the Lyapunov exponent and imaginary part the rotation number of the solution $\psi(x)$. The Lyapunov exponent vanishes in the stable bands, while the rotation number takes (a fixed multiple of) integer values in the gaps. In the classical field theory problem these bands of spectrum correspond precisely to the stability bands of the Mathieu equation (4) (for $q(t) = 0$).

Let us now turn out noise satisfying the conditions listed in the previous section and look for solutions of (6). Due to the translation invariance of the problem, we can take $x = 0$ to be the point at which ψ takes on its maximum. In the language of the quantum mechanics problem, the Lyapunov exponent can be extracted as follows:

$$\mu_E = -\lim_{x\to\infty}\frac{1}{x}log|\psi(x)|\,. \qquad (8)$$

Since the wave function in quantum mechanics must be normalized, no exponentially growing solutions are allowed. An eigenstate corresponds to a wave function which does not grow either for $x \to \infty$ or $x \to -\infty$. Thus, a positive real Lyapunov exponent corresponds to an exponentially decaying wave function, whereas an imaginary Floquet exponent with absolute value 1 corresponds to Bloch waves. A standard mathematical argument [8, 24–26] shows that the eigenstates are dense and the spectrum is pure point.

If we consider an energy eigenvalue E which lies in a conduction band (i.e. an energy band for which Bloch wave solutions exist) in the absence of noise, then our result (5) implies that as soon as a random noise term is added to the potential in (7), the Lyapunov exponent becomes positive, and that thus the corresponding wave function are exponentially decaying. This means that the addition of any noise satisfying the conditions listed in the previous section exponentially localizes the wave function. The localization length is inversely proportional to the Lyapunov exponent.

IV. DISCUSSION

We have shown that the time-independent Schrödinger equation for the non-relativistic Hamiltonian (7) does not admit Bloch wave solutions for any amplitude of the noise. Given a wave function centered at some point x (which we can without loss of generality take to be $x = 0$), we have shown that the wave function decays exponentially away from $x = 0$. Thus, we have given a new proof of the classic result [3] of exponential localization of states in one spatial dimension which extends the known result to the case of a periodic background potential.

Our methods are based on translating the time-independent quantum mechanical problem to a classical dynamical systems problem, and thus are only applicable to the case of one spatial dimension and cannot be used to study Anderson localization in a higher number of spatial dimensions.

The dynamical systems problem turns out to be a problem concerning the effects of noise on the resonant production of particles during the reheating phase of inflationary cosmology. In mathematical language, our problem concerns adding aperiodic noise to the mass term in the Mathieu equation. We have shown that the Lyapunov exponent increases for any amplitude of the noise. Thus, the solutions exhibit exponential behavior both forwards and backwards in time (starting from the

initial time $t = 0$ corresponding to $x = 0$ for the quantum problem). Normalizability of the wave function in the quantum problem implies that the eigenstates are precisely those solutions which decay away from the source point.

Acknowledgments

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[27] This is self-consistent if the period of energy transfer turns out to be short compared to the time scale of the expansion of space.

[28] Non-perturbatively, this was only shown for values of k in the stability bands. Perturbatively, the statement also holds for values of k in the instability bands [14]. What is relevant for this paper is the result for values of k in the stability bands.

[29] This toy model was already used in the first studies of reheating in inflationary cosmology [15, 16].

[30] The quantity g is a constant which has dimensions of mass. It is the coupling constant.

[31] The expansion of the Universe can be taken into account [17–19] without affecting the result that there are exponential instabilities.

[32] At some point, the back-reaction of the energy in the χ field on the dynamics of space-time will become important, and this could terminate the energy transfer [22].

[33] The mathematically much more complicated problem of inhomogeneous noise, in which case the inhomogeneity in the noise couples the different Fourier modes of χ, thus leading to a problem in the field of partial differential equations, was studied in [23].