Transformation theory of Hamiltonian PDE and the problem of water waves

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Abstract This set of lecture notes gives (i) a formal theory of Hamiltonian systems posed in infinite dimensions, (ii) a perturbation theory in the presence of a small parameter, adapted to reproduce some of the well-known formal computations of fluid mechanics, and (iii) a transformation theory of Hamiltonian systems and their symplectic structures. A series of examples is given, starting with a rather complete description of the problem of water waves, and, following a series of scaling and other simple transformations placed in the above context, a derivation of the well known equations of Boussinesq and Korteweg deVries.

1 Hamiltonian systems

A *Hamiltonian system* is given in terms of a Hamiltonian function $H : M \to \mathbb{R}$, where M is the phase space. We will restrict ourselves to phase spaces which are Hilbert spaces, denoting the inner product between two vectors $V_1, V_2 \in T(M)$ by $\langle V_1 | V_2 \rangle$. The symplectic structure is as usual given by a two-form ω on (M), which can be represented by the inner product, namely $\omega(V_1,V_2) = \langle V_1 | J^{-1} V_2 \rangle$, where, because of the antisymmetry of two-forms, the operator J satisfies $J^{-T} = -J^{-1}$. The Hamiltonian vector field X_H is defined through the relation $dH(V) = \omega(V, X_H)$ which is asked to hold for all $V \in T(M)$. The system of equations that we study, known as *Hamilton's canonical equations*, is given by

$$\dot{v} = X_H(v) \ , \qquad v(0) = v_0 \ . \tag{1}$$

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The inner product enters into the definition of the gradient of functions on M, which is in particular that for all $V \in T(M)$, $dH(V) = \langle grad_v H | V \rangle$, therefore Hamiltonian vector fields are expressed by

$$X_H = J grad_v H(v) \ . \tag{2}$$

We will denote the solution map, or the *flow*, for the initial value problem for system (1) by $v(t) = \varphi_t(v_0)$. From the usual theory of ordinary differential equations, whenever the Hamiltonian vector field $X_H(v)$ is $C^1(M, T(M))$ (usually meaning when the Hamiltonian $H(v)$ itself is $C^2(M, \mathbb{R})$) then the flow is defined and unique, at least locally in time. The disclaimer is that this regularity property holds very rarely the case when equation (1) describes a partial differential equation (the BBM equation is a notable exception), and much effort has gone into the study of the well posedness of the initial value problem and the properties of the solution map for numerous important examples of evolution equations. Furthermore, in this effort it is not clear that the property of being a Hamiltonian system is of particular importance in general. Nonetheless, because of its interest in various special cases, and because Hamiltonian partial differential equations (PDE) appear naturally in many areas of physics, it seems reasonable to take seriously the analogy between Hamiltonian dynamical systems and PDEs. This is one purpose of the presentation in this note.

2 Partial differential equations as Hamiltonian systems

It seems most useful to discuss Hamiltonian PDEs with a good set of examples. These are supplied by problems in physics, and in particular the ones I bear in mind most often come from the problems in wave propagation in fluid mechanics.

(i) The wave equation

Consider a scalar field $u(x,t)$ defined for $x \in \Omega \subseteq \mathbb{R}^d$ which satisfies the equation

$$\partial_t^2 u = \Delta_x u - g(u,x) \ , \qquad u(x,t) = 0 \ \text{ when } \ x \in \partial\Omega \ . \tag{3}$$

This can be written in the form of equation (1); indeed define

$$H(u,p) := \int_\Omega \tfrac{1}{2} p^2 + \tfrac{1}{2} |\nabla u|^2 + G(u,x) \, dx \ , \tag{4}$$

where $\partial_u G = g$. Then the second order equation (3) can be equivalently written as a first order system of PDEs

$$\begin{aligned} \dot{u} &= p = grad_p H \\ \dot{p} &= \Delta u - \partial_u G = -grad_u H \ . \end{aligned} \tag{5}$$

The gradient is taken with respect to the $L^2(\Omega)$ inner product, which dictates as well which Hilbert space we should propose for M. Actually, as operators such as Δ are unbounded, the initial value problem should normally be posed only on an appropriate subdomain of M. In any case, this problem is in the form of a Hamiltonian system with $v = (u, p)^T$ and

$$\dot{v} = J grad_v H \ , \qquad J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \ . \tag{6}$$

We will say that a Hamiltonian system with J of this form is in *Darboux coordinates*.

(ii) Burger's equation

A famous example in the theory of shoch waves is *Burger's equation*, which can be written in Hamiltonian form as well.

$$\partial_t w = w \partial_x w \qquad x \in \mathbb{R}^1 \ . \tag{7}$$

Define the Hamiltonian as

$$H := \int_{\mathbb{R}} \tfrac{1}{6} w^3 \, dx \ , \tag{8}$$

from which we compute the form of Hamiltonian's canonical equations

$$\dot{w} = \partial_x (\tfrac{1}{2} w^2) = J grad_{\,w} H \ , \qquad J := \partial_x \ . \tag{9}$$

Notice that the symplectic structure is given by an operator with no direct finite dimensional analog; it furthermore is not invertible, meaning that our formal discussion of the representation of the symplectic form in section 1 has to be taken with a grain of salt. It is well known that every nonconstant solution of Burger's equation develops discontinities, or *shocks*. The standard law of conservation of the Hamiltonian function, $H(\varphi_t(w)) = H(w)$ holds for smooth solutions, however it does not hold in most cases for time t after the time T of formation of a shock.

(iii) The Korteweg deVries equation

The classical Korteweg deVries (KdV) equation was derived as a model equation for the propagation of waves in the surface of a fluid. The beautiful fact about the KdV is that it is an example of an infinite dimensional completely integrable system, with algebraic integrals viewed in the proper coordinates. This integrability is not the topic of the present discussion. Rather, we show that it can be posed as a Hamiltonian PDE, and furthermore we discuss its relationship to fluid dynamics. The KdV equation for a function $r(X,t)$is normally written as

$$\partial_t r = -\tfrac{1}{6} \partial_X^3 r + 3 r \partial_X r \ . \tag{10}$$

This takes the form of a Hamiltonian system with Hamiltonian

$$H := \int_{\mathbb{R}} \tfrac{1}{12}(\partial_X r)^2 + \tfrac{1}{2} r^3 \, dX \ , \qquad J = \partial_X \ . \tag{11}$$

One easily checks that this is in the form (1), which in this context is

$$\dot{r} = \partial_X grad_{\,r} H \ . \tag{12}$$

The nonlinearity $g(X,r) = \partial_X(3r^2/2)$ is not the only one of interest. In particular the case $\partial_X r^3$ is a Hamiltonian PDE which is also a completely integrable system. Replacing either of the above two equations with a general nonlinear term $g(X,r)$ also results in a Hamiltonian PDE, which is sometimes considered as a model dispersive evolution equation which is not completely integrable.

(iv) The Boussinesq system

Another well known PDE which was originally derived in the study of water waves is the Boussinesq system,

$$\partial_t \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & \partial_X \\ \partial_X & 0 \end{pmatrix} \begin{pmatrix} p + \frac{1}{2} q^2 \\ qq + \partial_X^2 q + pq \end{pmatrix} \ . \tag{13}$$

This system of equations is a variant of a common one studied by Zakharov [13], and it has been shown to be another example of a completely integrable Hamiltonian PDE in Kaup [10] and Sachs [11]. The Hamiltonian for the system (13) is given by

$$H := \tfrac{1}{2} \int_{\mathbb{R}} p^2 + q^2 - (\partial_X q)^2 + pq^2 \, dX \ , \tag{14}$$

with a symplectic structure given by the matrix operator

$$J := \begin{pmatrix} 0 & \partial_X \\ \partial_X & 0 \end{pmatrix} \tag{15}$$

which is already in appearance in the above system of equations (13).

We now have a number of examples in hand, many of which stem originally from the study of water waves, that is the fluid dynamical problem of wave propagation in the surface of a body of fluid. A natural question is as to how these systems are related to each other. In particular we note that among these systems the number of dependent variables are different, the number of independent variables is different, and the symplectic structures are also changed from one system to another. In order to address this question, even on the formal level that is given in these lectures, we will undertake a detailed description of the problem of water waves itself from the point of view of the equation as an infinite dmensional Hamiltonian dynamical system.

3 The problem of water waves

The equations of evolution for the free surface of a body of water in the influence of gravity as a restoring force are a classical example of a system of Hamiltonian PDEs for which the structure of the equations as such has led to important developments in fluid dynamics. I will first describe the system of equations in standard Eulerian coordinates, after which the formulation of the problem as a Hamiltonian PDE can be derived. The fluid domain is given by $S(\eta) := \{x \in \mathbb{R}^{d-1}, y \in (-h, \eta(x))\}$, where we are assuming that the free surface is given as the graph of the function η; $\Gamma(\eta) := \{(x,y) : y = \eta(x)\}$. Normally the dimension is taken to be either $d = 2,3$, although mathematically it makes sense for it to be any integer $d \geq 2$. The force of gravity is take to act vertically, given by $F = -g(0,1)$. One of the unknowns of the problem is the time dependent fluid domain $S(\eta)$ defined in terms of the function $\eta(\cdot,t)$. The other unknowns are the components of the fluid velocity $\mathbf{u}(x,y,t)$ at every point in space and time in the fluid domain.

In $S(\eta)$ the fluid velocity vector field is taken to satisfy the conditions of incompressibility and irrotationality, respectively

$$\nabla \cdot \mathbf{u} = 0 \,, \qquad \nabla \wedge \mathbf{u} = 0 \,.$$

The latter is the condition that the vector field $\mathbf{u}$ is given in the form of a potential flow; $\mathbf{u} = \nabla\varphi$ at each instant of time, while the former states that the potential φ is harmonic in $S(\eta)$;

$$\Delta\varphi = 0 \,.$$

Furthermore, on the solid bottom boundary of $S(\eta)$ the fluid velocity is taken to have no normal component; $N \cdot \mathbf{u} = 0$, hence the potential satisfies Neumann boundary conditions on this component of the domain boundary;

$$N \cdot \nabla\varphi = 0 \,.$$

All of the time dependent and nonlinear content of the problem is thus expressed in the boundary conditions posed on the free surface $\Gamma(\eta)$, namely

$$\begin{aligned} \partial_t \eta &= \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi \\ \partial_t \varphi &= -g\eta - \tfrac{1}{2}|\nabla\varphi|^2 \,, \end{aligned} \tag{16}$$

known respectively as the kinematic and the Bernoulli conditions. The first boundery condition follows from the fact that a fluid particle which originates on the free surface will remain on the free surface under time evolution, so that a tangent vector T to its trajectory in space-time must always be orthogonal to the space-time normal vector N to the free surface; $N \cdot T = 0$. The Bernoulli condition simply represents an expression of the Euler equations for an inviscid fluid, in integrated form and evaluated on the free surface which itself is a surface of constant pressure.

The energy H of the system of equations for fluid motion with a free surface is straightforward to express, indeed it is the sum of kinetic and potential energy contributions;

$$H = K + P := \int_{\mathbb{R}^{d-1}} \int_{h}^{\eta(x)} |\mathbf{u}|^2 \, dy dx + \int_{\mathbb{R}^{d-1}} \int_{h}^{\eta(x)} gy \, dy dx \tag{17}$$

$$= \int_{\mathbb{R}^{d-1}} \int_{h}^{\eta(x)} \tfrac{1}{2} |\nabla \varphi|^2 \, dy dx + \int_{\mathbb{R}^{d-1}} \tfrac{g}{2} \eta^2 \, dx - C \, , \tag{18}$$

where the constant C is irrelevant to the dynamics and can be neglected. It is useful to rewrite the kinetic energy by integrating by parts.

$$K = \int_{\mathbb{R}^{d-1}} \int_{h}^{\eta(x)} \tfrac{1}{2} |\nabla \varphi|^2 \, dy dx = - \int_{\mathbb{R}^{d-1}} \int_{h}^{\eta(x)} \tfrac{1}{2} \varphi \Delta \varphi \, dy dx$$
$$+ \int_{\mathbb{R}^{d-1}} \tfrac{1}{2} \varphi N \cdot \nabla \varphi \, dS_{\mathrm{bottom}} + \int_{\mathbb{R}^{d-1}} \tfrac{1}{2} \varphi N \cdot \nabla \varphi \, dS_{\mathrm{free\,surface}} \, .$$

Because the velocity potential is harmonic and satisfies Neumann bottom boundary conditions, the first two terms of the right hand side vanish. Denoting the boundary values on the free surface $\Gamma(\eta)$ by $\xi(x) = \varphi(x, \eta(x))$, we have then

$$K = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2} \xi N \cdot \nabla \varphi \, dS_{\mathrm{free\,surface}} \, .$$

We are taking care to distinguish between φ the potential function itself, and ξ its values on the free surface $\Gamma(\eta)$. The elements of Laplace's equation that remain in this expression are the normal derivative of the potential φ on the free surface. It is useful to describe this quantity in terms of the boundary values $\xi(x)$ and an integral operator on the free surface itself.

Definition 1. *(Dirichlet – Neumann operator)* For the fluid domain $S(\eta)$ defined by the function $\eta \in C^1$, give boundary values $\xi(x)$ on the free surface $\Gamma(\eta)$, and consider their harmonic extension $\varphi(x, y)$ to the fluid domain satisfying Neumann bottom boundary conditions. The Dirichlet – Neumann operator is defined by the normal derivative of φ on the free surface, namely

$$G(\eta)\xi(x) = (\partial_y - \partial_x \eta(x) \cdot \partial_x) \varphi(x, \eta(x)) = R(N \cdot \nabla \varphi)(x, \eta(x)) \, , \tag{19}$$

where $R = \sqrt{1 + |\partial_x \eta|^2}$ is a normalization factor so that $G(\eta)$ is self-adjoint on $L^2(dx)$.

The Hamiltonian (17) can be conveniently written in terms of $G(\eta)$, indeed following [7] we write

$$H = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2} \xi G(\eta) \xi + \tfrac{g}{2} \eta^2 \, dx \, . \tag{20}$$

Theorem 1. (Zakharov [12]) *There exist canonical variables for the water waves problem* (16)*, in which it can be written in the form* (1) *in Darboux coordinates, with Hamiltonian* (20)*.*

Proof. Our derivation of the canonical conjugate variables is based on first principles of mechanics. Given the kinetic energy K and the potential energy P, the Lagrangian for the water waves problem is clearly

$$L = K - P \ . \tag{21}$$

We should express this in terms of the quantities $(\eta, \dot{\eta})$ (*i.e.* tangent space variables), for which we use the kinematic condition (16),

$$\dot{\eta} = \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi = G(\eta)\xi \ .$$

The Lagrangian is thus

$$L(\eta, \dot{\eta}) = \tfrac{1}{2} \int_{\mathbb{R}^{d-1}} \dot{\eta} G \eta) \dot{\eta} - \tfrac{g}{2} \eta^2 \, dx \ . \tag{22}$$

From this expression the Legendre transform dictates that the canonical conjugate variables are $(\eta, \partial_{\dot{\eta}} L) = (\eta, G^{-1}(\eta)\dot{\eta}) = (\eta, \xi)$. These are precisely the variables presented by Zakharov in [12], in terms of which one may give the water waves Hamiltonian (20).

□

Therefore the equations for water waves can be rewritten as a Hamiltonian system in Darboux coordinates;

$$\begin{aligned} \dot{\eta} &= grad_{\xi} H = G(\eta)\xi \\ \dot{\xi} &= -grad_{\eta} H = -g\eta - grad_{\eta} K \ . \end{aligned} \tag{23}$$

It is interesting to remark that the expressions for K and $grad_{\eta} K$ involve derivatives of the Dirichlet – Neumann operator with respect to perturbations of the domain $S(\eta)$. This idea was already discussed by Hadamard [8][9] in his Collège de France lectures in 1910 and 1916, in the context of the Green's function for Laplace's equation on a domain in $\mathbb{R}^d$. In these lectures he explicitly mentions the possibility of hydrodynamical applications.

4 The Dirichlet - Neumann operator

Any analysis of the water wave in the above formulation depends upon a detailed knowledge of the Dirichlet – Neumann operator. The fluid domain $S(\eta)$ is given by $\eta(x)$ defining the free surface. Given $\xi(x)$ the boundary values for the velocity potential, then $\varphi(x,y)$ is its harmonic extension to $S(\eta)$ which satisfies the appropriate Neumann bottom boundary conditions. The principal facts about $G(\eta)\xi(x) = \partial_y\varphi(x,\eta(x)) - \partial_x\eta \cdot \partial_x\varphi(x,\eta(x))$ that we use are contained in the lemma.

Proposition 1. *Suppose that $\eta \in C^1$. Then $G(\eta)$ satisfies the following properties:*

1. *$G(\eta)$ is positive semidefinite.*
2. *It is self-adjoint (on an appropriatly chosen domain).*
3. *$G(\eta)$ maps $H^1(\Gamma)$ to $L^2(\Gamma)$ continuously.*
4. *As an operator $G(\eta) : H^1(\Gamma) \to L^2(\Gamma)$ it depends analytically upon $\eta \in B_R(0) \subseteq C^1(\Gamma)$, for a nonzero value of R.*

The latter item entails questions of the bounded of singular integrals on hypersurfaces, and was proved in the case $d = 2$ by Coifman & Meyer [2], and in the case $d \geq 2$ in [6] using the fundamental results of Christ & Journé [1]. In particular it implies the existence of a convergent Taylor expansion for the operator.

Lemma 1. *The Taylor expansion of the Dirichlet – Neumann operator is given by the expression*

$$G(\eta)\xi = \sum_{j\geq 0} G^{(j)}(\eta)\xi \tag{24}$$

where each $G^{(j)}(\eta)$ is homogeneous of degree j in η. Explicitely,

$$G^{(0)}\xi(x) = |D_x|\tanh(h|D_x|)\xi(x) \tag{25}$$

$$G^{(1)}(\eta)\xi(x) = D_x \cdot \eta D_x - G^{(0)}\eta G^{(0)}\xi(x) \tag{26}$$

$$G^{(2)}(\eta)\xi(x) = \tfrac{1}{2}(G^{(0)}\eta^2 D_x^2 + D_x^2\eta^2 G^{(0)} - 2G^{(0)}\eta G^{(0)}\eta G^{(0)})\xi(x)\ . \tag{27}$$

The terms $G^{(j)}(\eta)$ in the Taylor expansion are polynomial expressions in the quantities D_x and $G^{(0)}$ of order $j+1$, however for $\eta \in C^1$ these terms are nevertheless bounded from $H^1 \to L^2$. I is because of the form of the operator which is related to a multiple commutator; $[\eta,\dots j\times\dots[\eta,D_x^j]] = (-1)^j j!(\partial_x\eta)^j$. With regard to this series for the Dirichlet – Neumann operator, the water waves Hamiltonian itself is analytic on an appropriately chosen subset of , and possesses a Taylor series expansion about the equilibrium solution $(\eta,\xi) = 0$, namely

$$\begin{aligned} H(\eta,\xi) &= \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi G^{(0)}\xi + \tfrac{g}{2}\eta^2\,dx + \sum_{j\geq 3}\tfrac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G^{(j-2)}(\eta)\xi\,dx \\ &= \sum_{j\geq 2} H^{(j)}(\eta,\xi)\ , \end{aligned} \tag{28}$$

where $H^{(j)}(\eta,\xi)$ is homogeneous of degree j with respect to the variables (η,ξ).

5 Perturbation theory

Suppose that the Hamiltonian function H depends upon an additional parameter ε; $H(v;\varepsilon) = H^{(0)} + \varepsilon H^{(1)} + \dots \varepsilon^m H^{(m)} + \varepsilon^{m+1} R(v;\varepsilon)$, for $\varepsilon \in \mathscr{E}$ a space of parameters. It is natural to approximate orbits $v(t;\varepsilon)$ by those of the truncated problem

$$\dot{v} = J grad_v (H^{(0)} + \varepsilon H^{(1)} + \dots \varepsilon^m H^{(m)}) \qquad (29)$$
$$v(0) = v_0 \, , \qquad v(t) = v(t;\varepsilon, m)$$

The solution $v(t) = v(t;\varepsilon, m)$ clearly depends upon both ε and the degree m of the Taylor series approximation, and there is the natural expectation that, at least for finite time intervals, the solutions $v(t;\varepsilon, m)$ of (29) approximate the solutions of the full problem (1), with a better approximation given with larger m. Indeed, for C^2 Hamiltonians this is the case.

Proposition 2. *Suppose that the Hamiltonian $H \in C^{2,m+1}(M \times \mathscr{E})$. Then, at least for bounded time intervals $|t| \leq T_0$, approximate orbits $v(t;\varepsilon, m)$ of* (29) *are ε^m close to orbits of the full Hamiltonian system* (1).

Our intentions are to discuss Hamiltonian systems in infinite dimensional Hilbert spaces, and in particular Hamiltonian partial differential equations, which we have already pointed out are rarely given by smooth Hamiltonian vector fields. Therefore the above proposition is not applicable. Nonetheless it serves as a basic guiding principle to the problems we are aiming to discuss. It is also true that one can often do better than Proposition 2, and in some cases the length of the time interval of validity of this approximation may be longer, or indeed very much longer. However the only improvement on this statement that can be made at this level of generality is that, if the Lyapunov exponents of both (1)(29) are bounded, then for any $m' < m$, approximating orbits remain $\varepsilon^{m'}$ close to true orbits for times $|t| \leq T_\varepsilon$, with $T_\varepsilon \sim \log(1/\varepsilon)$.

6 The calculus of transformations

Given a Hamiltonian system

$$\dot{v} = J grad_v H \qquad (30)$$

posed on a phase space M, we will subject it to transformations of variables of M. Consider two phase spaces M_1 and M_2 with a symplectic form on M_1 given in terms of J_1. Let $H_1 : M_1 \to \mathbb{R}$ be a Hamiltonian. A transformation

$$\tau : M_1 \to M_2 \, , \qquad v \mapsto w = \tau(v) \qquad (31)$$

gives rise to a Hamiltonian defined on M_2, namely $H_2(w) = H_2(\tau(v)) = H_1(v)$. The Hamiltonian vector field $X_{H_1} = J_1 grad_v H_1$ is transformed as follows;

$$\dot{w} = \partial_v \tau(v)\dot{v} = \partial_v \tau(v) J_1 grad_{\,v} H_1(v) \ ,$$

while on the other hand

$$grad_{\,v} H_1(v) = (\partial_v \tau)^T grad_{\,w} H_2(\tau(v)) \ .$$

Equating the expressions, one observes the following:

Proposition 3. *The vector field $X_{H_1} = J_1 grad_{\,v} H_1$ is transformed to*

$$\dot{w} = \partial_v \tau(v) J_1 (\partial_v \tau)^T grad_{\,w} H_2(\tau(v)) \ . \tag{32}$$

We denote $J = \partial_v \tau(v) J_1 (\partial_v \tau)^T$ which can be used to define a symplectic structure on M_2. When M_2 already has a symplectic structuree represented by J_2, and the transformation $w = \tau(v)$ is such that $J_2 = \partial_v \tau(v) J_1 (\partial_v \tau)^T$, then τ is called *canonical*. In partcular when $M_1 = M_2$ and $J_1 = J_2$ is given in Darboux coordinates, these are the usual canonical transformations which play a special rôle in the subject of Hamiltonian mechanics.

Examples of transformations. While the subject of canonical transformations and their generating functions is basic knowledge in finite dimensional Hamiltonian systems, it is less developed in the study of PDE and other infinite dimensional cases. In the following paragraphs we will work through some of the more elementary transformations that occur in Hamiltonian PDE, putting them into context. Furthermore we will make use of particular parameter families of such transformations in order to introduce a small parameter into the Hamiltonian. In this way the principle outlined in section 5 can be invoked, with the result that we have a natural approximation procedure for solutions through a (albeit formal) series expansion of the Hamiltonian. This procedure and its general context has been worked out in a number of papers that have appeared over the span of several years, by the author along with M. Groves [3], P. Guyenne & H. Kalisch [4] and P. Guyenne, D. Nicholls & C. Sulem [5].

Initially, the setting is that $M = L^2(\mathbb{R}^{d-1})^2$ will be considered the phase space, with

$$v = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \in M \ , \qquad \langle v_1 | v_2 \rangle = \int_{\mathbb{R}^{d-1}} \eta_1 \eta_1 + \xi_1 \xi_2 \, dx \tag{33}$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{34}$$

which is the case of Darboux coordinates.

(i) Amplitude scaling.

Consider the elementary transformations $\tau : v \mapsto w$, where

$$w = \begin{pmatrix} \eta' \\ \xi' \end{pmatrix} = \begin{pmatrix} \alpha\eta \\ \beta\xi \end{pmatrix} = \tau(v) \;, \tag{35}$$

for $\alpha, \beta \in \mathbb{R}^+$. The Jacobian of the transformation τ is given by

$$\partial_v \tau = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}$$

therefore the symplectic form induced by the transformation is

$$J_1 = \partial_v \tau J \partial_v \tau^T = \alpha\beta J \;, \tag{36}$$

with the Darboux operator J given in (34). The effects of such transformations are easily restored to the usual Darboux coordinates through a time change $t' = \alpha^{-1}\beta{-}1t$.

The small amplitude regime of the water wave problem is introduced by an amplitude scaling which is a transformation of this form. Namely one sets

$$\begin{pmatrix} \varepsilon^2\eta' \\ \varepsilon\xi' \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \;, \qquad \varepsilon << 1 \;, \tag{37}$$

which is to say that we are seeking solutions for which the amplitude η of a solution is small, and represented in its asymptotic regime by an order one quantity η' times ε^2, and similarly for $\xi = \varepsilon\xi'$. The resulting change of symplectic form is that

$$J_1 = \varepsilon^{-3} J \;,$$

which is equivalent to a rescaling to a slow time variable. The effect on the water waves Hamiltonian (20) and its Taylor series expansion (28) is that

$$H_1 = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\varepsilon^2 \xi' G^{(0)} \xi' + \tfrac{g}{2}\varepsilon^4 \eta'^2 \, dx + \sum_{j\geq 3} \tfrac{1}{2} \int_{\mathbb{R}^{d-1}} \varepsilon^{2+2j} \xi' G^{(j-2)}(\eta')\xi' \, dx \;.$$

In particular a small parameter has been introduced into the Hamiltonian $H_1 = H_1(\eta', \xi'; \varepsilon)$, for which one may consider approximations by its Taylor series. For instance, up to order $\mathscr{O}(\varepsilon^4)$

$$\varepsilon^2 H_1^{(2)} + \varepsilon^4 H_1^{(4)} = \varepsilon^2 \Big(\int \tfrac{1}{2} \xi' G^{(0)} \xi' \, dx \Big) + \varepsilon^4 \Big(\int \tfrac{g}{2} \eta'^2 + \tfrac{1}{2} \xi' G^{(1)}(\eta') \xi' \, dx \Big) \;,$$

where we recall that $G^{(1)}(\eta') = D_x \eta' D_x - G^{(0)} \eta' G^{(0)}$.

(ii) Spatial scaling.

The long wave regime of the water waves problem highlights solutions whose typical wavelength is asymptotically long; it is represented through a small parameter introduced into the problem by the spational scaling

$$x \mapsto X := \varepsilon x \; . \tag{38}$$

The resulting transformation of phase space M is thus

$$\tau : v(x) \mapsto w(X) = v(X/\varepsilon) = \tau(v)(X) \; . \tag{39}$$

The Jacobian of the transformation on a vector field $V(x) \in T_v(M)$ is

$$\partial_v \tau(v)V(X) = \frac{d}{ds}|_{s=0}\Big(v(X/\varepsilon) + sV(X/\varepsilon) \Big) = V(X/\varepsilon) \; .$$

The transpose is slightly less obvious, we compute it using the identity;

$$\langle V_1 | \partial_v \tau V_2 \rangle = \int_{\mathbb{R}^{d-1}} V_1(X) V_2(X/\varepsilon), dX \tag{40}$$

$$= \int_{\mathbb{R}^{d-1}} V_1(\varepsilon x) V_2(x)\, \varepsilon^{d-1} dx = \langle (\partial_v \tau)^T V_1 | V_2 \rangle \; . \tag{41}$$

Therefore $(\partial_v \tau)^T V(x) = \varepsilon^{d-1} V(\varepsilon x)$, and the induced symplectic form is

$$J_2 = \partial_v \tau J (\partial_v \tau)^T = \varepsilon J \; , \tag{42}$$

at least if we are working with the Darboux symplectic structure. Thus, modulo a rescaling of time, this recovers the original symplectic form.

It is necessary to study the effect that this transformation has on the Hamiltonian.

Lemma 2. *Let $\tau(v)(X) = v(X/\varepsilon) = w(X)$ be be the transformation in question, and let $m(D_x)$ be a Fourier multiplier oprerator*

$$(m(D_x)v)(x) = \frac{1}{(2\pi)^{d-1}} \int\!\!\int_{\mathbb{R}^{2(d-1)}} e^{ik\cdot(x-x')} m(k) v(x')\, dx' dk \; . \tag{43}$$

Under τ, the operator is transformed to

$$\tau(m(D_x)v)(X) = (m(\varepsilon D_X)\tau(v))(X) \; . \tag{44}$$

Proof. This is the fact that that cotangent variables (x,k) of pseudo-differential operators are transformed symplectically under changes of variables. Indeed one calculates

$$\begin{aligned}\tau(m(D_x)v)(X) &= \frac{1}{(2\pi)^{d-1}}\int\int_{\mathbb{R}^{2(d-1)}} e^{ik\cdot(X/\varepsilon - x')}m(k)v(x')\,dx'dk \\ &= \frac{1}{(2\pi)^{d-1}}\int\int_{\mathbb{R}^{2(d-1)}} e^{ik\cdot(X/\varepsilon - X'/\varepsilon)}m(k)v(X'/\varepsilon)\,\frac{dX'dk}{\varepsilon^{d-1}} \\ &= \frac{1}{(2\pi)^{d-1}}\int\int_{\mathbb{R}^{2(d-1)}} e^{iK\cdot(X - X')}m(\varepsilon K)v(X'/\varepsilon)\,dX'dK \\ &= m(\varepsilon D_X)\tau(v)(X)\ .\end{aligned}$$

□

Considering the water wave Hamiltonian, the Dirichlet – Neumann operator

$$G^{(0)}(D_x) = |D_x|\tanh(h|D_x|)$$

is transformed to

$$G^{(0)}(\varepsilon D_X) = \varepsilon|D_x|\tanh(\varepsilon h|D_x|) \sim \varepsilon^2 h|D_X|^2 - \frac{\varepsilon^4 h^3}{3}|D_X|^4 + \dots \tag{45}$$

Using this expression, the Hamiltonian (20) becomes

$$\begin{aligned}H_2 = \varepsilon^4\int_{\mathbb{R}^{d-1}}\Big(\tfrac{1}{2}\xi(h|D_X|^2\xi + \tfrac{g}{2}\eta^2\Big) + \frac{\varepsilon^2}{2}\Big(\xi(-\frac{h^3}{3}|D_X|^4\xi) + \xi D_X\cdot\eta D_X\xi\Big)\frac{dX}{\varepsilon^{d-1}} \\ +\varepsilon^7 R_2^{(2)}\ .\end{aligned} \tag{46}$$

(iii) Surface elevation - velocity coordinates.

It is often convenient to write the Euler equations in terms of the variables $w = (\eta, u)$, $u = \partial_x\xi$ instead of $v = (\eta, \xi)$. That is, the second variable represents a velocity instead of a potential; in this case it essentially represents the horizontal velocity of the fluid at the free surface $\Gamma(\eta)$. We restrict our discussion of these *surface elevation - velocity* coordinates to the case of two dimensions, for simplicity. That is,

$$w = (\eta, u) = \tau(v) = (\eta, \partial_x\xi)\ . \tag{47}$$

The Jacobian of the transformation is

$$\partial_v\tau(v) = \begin{pmatrix} I & 0 \\ 0 & \partial_x\end{pmatrix}$$

and the induced symplectic form is represented by the operator

$$J_2 = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0\end{pmatrix} = \partial_v\tau J(\partial_v\tau)^T\ , \tag{48}$$

where J is in Darboux coordinates. One recognizes this as the operator representing the Boussinesq symplectic form (15), up to a trivial change of sign.

Returning to the Hamiltonian (46), and phrasing it in surface elevation - velocity coordinates, we have

$$H_2 = \varepsilon^3 \int_{\mathbb{R}} \left(\tfrac{h}{2}u^2 + \tfrac{g}{2}\eta^2\right) + \tfrac{\varepsilon^2}{2}\left(\tfrac{-h^3}{3}(\partial_X u)^2 + \eta u^2\right) dX + \mathscr{O}(\varepsilon^7) \ , \qquad (49)$$

while using the operator J_2 of (48) when expressing Hamilton's equations (1). The truncated system (49) up to order $\mathscr{O}(\varepsilon^5)$ is precisely the Boussinesq system (13) (modulo adjusting the value of several constants and the sign change $(p,q)^T = (\eta, -u)^T$).

(iv) Moving reference frame.

It is part of the theory of nonlinear waves towork in coordinate systems which move with the characteristic speed of solutions, namely

$$v'(x,t) := v(x - tc, t) \ , \qquad (50)$$

for appropriate choices for the velocity c. However this transformation does not at first glance fit into the setting of the transformation theory described above, as the time variable is distinguished, and (50) mixes the rôles of the spatial and temporal variables. We observe however that in the problems under discussion the *momentum*

$$I(\eta, \xi) := \int_{\mathbb{R}} \eta(x)\partial_x \xi(x)\, dx \qquad (51)$$

is a conserved quantity, as can be seen from its Poisson bracket with the Hamiltonian

$$\{I, H\} := \langle \mathrm{grad}_v I | J\, \mathrm{grad}_v H\rangle = 0 \ . \qquad (52)$$

Therefore their respective flows commute; $\varphi_t^H \circ \varphi_s^I(v) = \varphi_s^I \circ \varphi_t^H(v)$. The Hamiltonian flow of the momentum

$$\partial_s \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathrm{grad}_v I = \begin{pmatrix} -\partial_x \eta \\ -\partial_x \xi \end{pmatrix} \qquad (53)$$

is simply constant unit speed translation

$$\varphi_s^I(v)(x) = v(x - s) \ .$$

Thus the flow along the diagonal is clearly $\varphi_t^H \circ \varphi_{tc}^I(v) = \varphi_t^{H+cI}$. Therefore the Hamiltonian flow of $H(v) + cI(v)$ is the Hamiltonian flow of $H(v)$ observed in a coordinate frame translating with velocity c.

In the context of the water wave problem the characteristic velocity is $c_0 := \sqrt{gh}$; to study the problem of water waves in our present point of view, we are to look at the flow of the system whose Hamiltonian is $H_2 + \sqrt{gh}I$.

Writing the momentum in surface elevation - velocity coordinates and scaling the coordinates appropriately, we find that

$$I = \varepsilon^3 \int_{\mathbb{R}} u\eta \, dX \ , \tag{54}$$

and therefore

$$\begin{aligned} H_2 + \sqrt{gh}I &= \varepsilon^3 \int_{\mathbb{R}} \tfrac{1}{2}\Big(hu^2 + 2\sqrt{gh}u\eta + g\eta^2\Big) + \tfrac{\varepsilon^2}{2}\Big(\tfrac{-h^3}{3}(\partial_X u)^2 + \eta u^2\Big) dX \\ &= \varepsilon^3 \int_{\mathbb{R}} \tfrac{1}{2}(\sqrt{h}u + \sqrt{g}\eta)^2 + \tfrac{\varepsilon^2}{2}\Big(\tfrac{-h^3}{3}(\partial_X u)^2 + \eta u^2\Big) dX \ . \end{aligned} \tag{55}$$

(v) Characteristic coordinates.

Focusing on the first term H_2 of the Hamiltonian, it is a common situation to have it in the quadratic form

$$H_2^{(2)} = \tfrac{1}{2} \int_{\mathbb{R}} Au^2 + B\eta^2 \, dX \ ,$$

with $A, B > 0$. Hamilton's equations (1) for $H_2^{(2)}$ alone are the wave equation

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix} grad_{\,v} H_2^{(2)} = \begin{pmatrix} 0 & -A \\ -B & 0 \end{pmatrix} \begin{pmatrix} \partial_X \eta \\ \partial_X u \end{pmatrix} \ . \tag{56}$$

We seek a transformation of coordinates $(r,s)^T = \tau(\eta,u)^T$ which will accomplish three things.
(1) It should diagonalize the symplectic form J_2;

$$J_3 := \partial_v \tau \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix} (\partial_v \tau)^T = \begin{pmatrix} \partial_X & 0 \\ 0 & -\partial_X \end{pmatrix} \ . \tag{57}$$

(2) It should transform the Hamiltonian to normal form

$$H_3^{(2)} = \tfrac{1}{2} \int_{\mathbb{R}} \sqrt{AB}(r^2 + s^2) \, dX \ . \tag{58}$$

(3) And it should transform the wave equation (56) to characteristic form

$$\partial_t \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \begin{pmatrix} \partial_X r \\ \partial_X s \end{pmatrix} \ . \tag{59}$$

The transformation $w = \tau(v) = Tv$, where T is the matrix

$$T = \begin{pmatrix} \sqrt[4]{\frac{B}{4A}} & -\sqrt[4]{\frac{A}{4B}} \\ \sqrt[4]{\frac{B}{4A}} & \sqrt[4]{\frac{A}{4B}} \end{pmatrix}$$

accomplishes all three of these goals, with the result that $C = \sqrt{AB}$.

In the case of the water wave Hamiltonian H_3, we have $A = h$ and $B = g$, so that

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \sqrt[4]{\frac{g}{4h}} & -\sqrt[4]{\frac{h}{4g}} \\ \sqrt[4]{\frac{g}{4h}} & \sqrt[4]{\frac{h}{4g}} \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix} , \tag{60}$$

and in these terms, the relevant Hamiltonian approximation which is to be valid up to $\mathscr{O}(\varepsilon^5)$ is given by

$$\begin{aligned} H_2 + \sqrt{gh} I = \varepsilon^3 \int_{\mathbb{R}} \sqrt{gh} s^2 \, dX \\ + \varepsilon^5 \int_{\mathbb{R}} -\frac{h^3}{6} \left(\sqrt{\frac{g}{4h}} \right) (\partial_X r - \partial_x s)^2 + \frac{1}{4\sqrt{2}} \sqrt[4]{\frac{g}{h}} (r^3 - r^2 s - r s^2 + s^3) \, dX \ . \end{aligned} \tag{61}$$

Now restrict this Hamiltonian to the hypersurface $M_1 := \{s = 0\} \subseteq M$, denoting it by H_4;

$$H_4 = \varepsilon^5 \int_{\mathbb{R}} -\frac{h^3}{6} \left(\sqrt{\frac{g}{4h}} \right) (\partial_X r)^2 + \frac{1}{4\sqrt{2}} \sqrt[4]{\frac{g}{h}} r^3 \, dX \ . \tag{62}$$

The subspace M_1 is a *symplectic* subspace of M, possessing the symplectic form $J_4 = \partial_X$, it being the restriction of the symplectic form J_3 of (57). This is unlike the situation in Darboux coordinates, in which M_1 would be a Lagrangian subspace. The equations of motion (1) for r on M_1, or at least in an $o(\varepsilon^2)$ neighborhood of it, are thus

$$\begin{aligned} \partial_t r &= \partial_X grad_r H_4 \\ &= \varepsilon^2 \partial_X (-c_1 \partial_X^2 r + c_2 r^2) \ , \end{aligned} \tag{63}$$

with $c_1 = \frac{h^3}{3} \sqrt{\frac{g}{4h}}$ and $c_2 = \frac{3}{4\sqrt{2}} \sqrt[4]{\frac{g}{h}}$. This is precisely the KdV equation given in (10), modulo a simple change of time scale $\partial_t = \varepsilon^2 \partial_\tau$ ($\tau = \varepsilon^2 t$), and with a few extra but unimportant constants that could have been normalized in the above calculation with some forethought.

We have seen that a formal calculation, using basic transformations and a small parameter have given us the KdV equation as an approximation of the equations of water waves. It has been a research program to understand the rigorous aspects of this correspondence between solutions of the full Euler's equations and solutions of the KdV or of other model long wave equations. However at this point none of the rigorous results follow along the lines of the above concatenation of transformations. We believe that such an approach would be a rewarding line of work.

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