Invariant tori for Hamiltonian PDE

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Hamiltonian PDE

- Flow in *phase space*, where $v \in \mathcal{H}$ a Hilbert space

$$\partial_t v = J \mathrm{grad}_v H(v) \ , \quad v(x,0) = v^0(x) \ , \tag{1}$$

- Symplectic form

$$\omega(X,Y) = \langle X, J^{-1} Y \rangle_{\mathcal{H}} \ , \quad J^T = -J \ .$$

- The flow $v(x,t) = \varphi_t(v^0(x))$
- Interested in orbits where

$$\overline{\{\varphi_t(v^0) \ : \ t \in \mathbb{R}\}} = \mathbb{T}^m$$

an m-dimensional torus. This gives stable motions of (1).

Outline

Hamiltonian systems

- A finite dimensional Hamiltonian system, $H(q, p) : \mathbb{R}^{2n} \mapsto \mathbb{R}$

$$\begin{aligned} \dot{q} &= \mathrm{grad}_p H \, , \\ \dot{p} &= -\mathrm{grad}_q H \end{aligned} \tag{2}$$

- Ask that $\mathrm{grad}\ H(0) = 0$ and $\mathrm{hess}\ H(0) > 0$,

$$H = H^{(2)} + R$$

After a change of variables,

$$\begin{aligned} H^{(2)} &= \frac{1}{2}|p|^2 + \frac{1}{2}\langle q, Aq\rangle \\ A &= \mathrm{diag}_{k=1\ldots n}(\omega_k^2) \end{aligned}$$

The harmonic oscillator

- Linearized problem about $(q,p) = 0$

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \partial_q H^{(2)} \\ \partial_p H^{(2)} \end{pmatrix} \qquad (3)$$

- The linear flow, setting $\xi_k(t) = t\omega_k$

$$\begin{aligned} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} &= \Phi_t \begin{pmatrix} q^0 \\ p^0 \end{pmatrix} \\ &= \mathrm{diag}_{2\times 2} \begin{pmatrix} \cos(\xi_k(t)) & \sin(\xi_k(t))/\omega_k \\ -\omega_k \sin(\xi_k(t)) & \cos(\xi_k(t)) \end{pmatrix} \begin{pmatrix} q^0 \\ p^0 \end{pmatrix} \end{aligned}$$

- Solutions lie on tori of dimension $m = \dim_{\mathbb{Q}} \{\omega_1, \dots \omega_n\}$.

$$\mathbb{T}^m = \overline{\{\Phi_t(q^0, p^0) \ : \ t \in \mathbb{R}\}}$$

The nonlinear wave equation

- On a domain $\mathbb{T}^d = \mathbb{R}^d/\Gamma$, for period lattice Γ

$$\partial_t^2 u - \Delta u + g(x,u) = 0\ . \qquad (4)$$

(Alternatively, $u = 0$ on the boundary of a domain $D \subseteq \mathbb{R}^d$).

- The Energy is

$$H(u,p) = \int_{\mathbb{T}^d} \tfrac{1}{2}p^2 + \tfrac{1}{2}|\nabla u|^2 + G(x,u)\, dx\ ,$$

- Equation (4) can be rewritten as

$$\begin{aligned} \partial_t u &= \mathrm{grad}_p H(u,p) &= p \\ \partial_t p &= -\mathrm{grad}_u H(u,p) &= \Delta u - \partial_u G(x,u)\ , \end{aligned}$$

in Darboux coordinates, where $g(x,\cdot) = \partial_u G(x,\cdot)$.

Linearized wave equation

- Suppose the Taylor series for $G(x, u)$ is

$$G(x, u) = \frac{1}{2} g_1(x) u^2 + \frac{1}{3} g_2(x) u^3 + \dots$$

- Then the Hamiltonian takes the form $H = H^{(2)} + R$, with

$$\begin{aligned} H^{(2)} &= \int_{\mathbb{T}^d} \tfrac{1}{2} p^2 + \tfrac{1}{2} |\nabla u|^2 + \frac{1}{2} g_1(x) u^2 \, dx \\ &= \sum_{k \in \Gamma'} \tfrac{1}{2} |\hat{p}_k|^2 + \tfrac{1}{2} \omega_k^2 |\hat{u}_k|^2 \end{aligned}$$

- Eigenfunction/eigenvalue pairs $(\psi_k(x), \omega_k^2)$ for the operator

$$L(g_1)\psi_k = (-\Delta + g_1(x))\psi_k = \omega_k^2 \psi_k \, .$$

The linearized flow

- Solutions of the linear wave equations are

$$\begin{pmatrix} u(x,t) \\ p(x,t) \end{pmatrix} = \Phi_t \begin{pmatrix} u^0(x) \\ p^0(x) \end{pmatrix}$$
$$= \sum_{k\in\Gamma'} \psi_k(x) \begin{pmatrix} \cos(\xi_k(t)) & \sin(\xi_k(t))/\omega_k \\ -\omega_k \sin(\xi_k(t)) & \cos(\xi_k(t)) \end{pmatrix} \begin{pmatrix} \hat{u}_k^0 \\ \hat{p}_k^0 \end{pmatrix}$$

- This is the harmonic oscillator with frequencies $\{\omega_k\}_{k\in\Gamma'}$.
- Generically, $\dim_{\mathbb{Q}} \{\omega_1, \dots\} = \infty$.

Basic facts

Some basic facts about the flow of the linearized problem:

- The Hamiltonian is preserved by the flow

$$H^{(2)}(\Phi_t(v)) = H^{(2)}(v)$$

  Conservation of energy

- All of the actions are preserved

$$I_k(v) := \frac{1}{2}(\omega_k|\hat{u}_k|^2 + \frac{1}{\omega_k}|\hat{p}_k|^2) = I_k(\Phi_t(v))$$

  The *moment map*: $(\hat{u}, \hat{p}) \mapsto I$

- The phases evolve linearly in time; $t \mapsto \{\xi_k(t) = \omega_k t\}_{k\in\Gamma'}$. Solutions are *periodic* when $\dim_{\mathbb{Q}} \{\omega_{j_1}, \omega_{j_2} \dots\} = 1$, *quasi-periodic* when $< +\infty$ and otherwise *almost-periodic*.

Basic questions

Basic questions regarding the flow of the nonlinear systems

$$\partial_t v = J \operatorname{grad}_v \left(H^{(2)} + R\right) . \tag{5}$$

(1) Whether *some* solutions are

- periodic $\mathbb{T}^1 \mapsto \mathcal{H}$
- quasi-periodic $\mathbb{T}^m \mapsto \mathcal{H}, \ m < +\infty$
- almost-periodic (and even Lagrangian invariant tori).

corresponding to stable motions of (5).

Basic questions

(2) Given data $v^0 \in \mathcal{H}$, whether

- The flow $\varphi_t(v^0)$ remains in $\mathcal{H}$ for all time (global well-posedness of the PDE),
- for $v^0 \in B_\varepsilon(0)$ then $\varphi_t(v^0) \in B_\delta(0)$ for all $t \in \mathbb{R}$ (stability),
- action variables change by controlled amounts

$$|I_k(\varphi_t(v)) - I_k(v)| < \varepsilon^\alpha$$

for $|t| < T(\varepsilon) \sim \exp 1/\varepsilon^\beta$ (Nekhoroshev stability).

Basic questions

(3) Whether for some solutions there are lower bounds on the growth of the action variables $I_k(\varphi_t(v^0)) - I_k(v^0)$, or of Sobolev norms for large $|t| >> 1$

$$\|\varphi_t(v^0)\|_{H^s} \geq \delta(t) \ , \quad s >> 1 \qquad \text{(Arnold diffusion)}.$$

Further examples

- **Nonlinear Schrödinger equation**

$$i\partial_t u - \frac{1}{2}\Delta_x u + Q(x,u,\overline{u}) = 0 \ , \quad x \in \mathbb{T}^d \tag{6}$$

with Hamiltonian

$$H_{NLS}(u) = \int_{\mathbb{T}^d} \frac{1}{2}|\nabla u|^2 + G(x,u,\overline{u})\, dx \ , \quad \partial_{\overline{u}} G = Q \ .$$

Rewritten

$$\partial_t u = i \, \mathrm{grad}_{\overline{u}} H_{NLS}$$

Further examples

- **Korteweg – de Vries equation**

$$\partial_t q = \frac{1}{6}\partial_x^3 q - \partial_x(\partial_q G(x,q)) \ , \quad x \in \mathbb{T}^1 \tag{7}$$

The Hamiltonian is

$$H_{KdV}(q) = \int_{\mathbb{T}^1} \frac{1}{12}(\partial_x q)^2 + G(x,q)\, dx$$

Rewritten

$$\partial_t q = J \operatorname{grad}_q H_{KdV} \ , \quad \text{where} \quad J = -\partial_x$$

Further examples

- **Surface water waves:** The fluid domain

$$\Sigma(\eta) = \{y = \eta(x,t) : x \in \mathbb{T}^{d-1}\}$$

Velocity field, a potential flow $\mathbf{u} = \nabla\varphi$, $\Delta\varphi = 0$.
Hamilton's principle

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \mathrm{grad}_\eta H(\eta,\xi) \\ \mathrm{grad}_\xi H(\eta,\xi) \end{pmatrix} , \tag{8}$$

Canonical conjugate variables:
$\eta(x)$ (free surface elevation)
$\xi(x) = \varphi(x, \eta(x))$
(boundary values of the velocity potential)

Surface water waves

- The Hamiltonian is

$$H(\eta,\xi) = \int \frac{1}{2}\xi G(\eta)\xi + \frac{1}{2}g\eta^2 \, dx \; ,$$

  where $G(\eta)$ is the Dirichlet – Neumann operator on $\Sigma(\eta)$.

- Taylor series of $G(\eta) = \sum_{j\geq 0} G^{(j)}(\eta)$

$$G^{(0)} = D\tanh(hD) \; , \qquad D := -i\partial_x$$
$$G^{(1)}(\eta) = D\eta(x)D - G^{(0)}\eta(x)G^{(0)}$$

  Hadamard's *variational principle*

Interactions of solitary waves

- Interaction of a head-on collision of two solitary waves of amplitudes $S/h = 0.4$

(a) (b)

- Long time behavior after collision

An invariant torus

- Mapping of a torus $S(\xi) : \mathbb{T}^m \mapsto \mathcal{H}$
- Flow invariance $S(\xi + t\Omega) = \varphi_t(S(\xi))$
  Frequency vector $\Omega \in \mathbb{R}^m$.
- This implies that both

$$\partial_t S = J \operatorname{grad}_v H(S) \ , \quad \text{and} \quad \partial_t S = \Omega \cdot \partial_\xi S \ . \tag{9}$$

- Problem: Solve (9) for $(S(\xi), \Omega)$.
  This is generally a small divisor problem.

  Rewrite (9) as

$$J^{-1}\Omega \cdot \partial_\xi S - \operatorname{grad}_v H(S) = 0 \ . \tag{10}$$

A variational formulation

Consider the space of mappings $S \in X := \{S(\xi) : \mathbb{T}^m \mapsto \mathcal{H}\}$.

- Define action functionals

$$
\begin{aligned}
I_j(S) &= \frac{1}{2} \int_{\mathbb{T}^m} \langle S, J^{-1} \partial_{\xi_j} S \rangle \, d\xi \\
\delta_S I_j &= J^{-1} \partial_{\xi_j} S
\end{aligned}
$$

The moment map for *mappings*

- The average Hamiltonian

$$
\begin{aligned}
\overline{H}(S) &= \int_{\mathbb{T}^m} H(S(\xi)) \, d\xi \\
\delta_S \overline{H} &= \mathrm{grad}_v H(S)
\end{aligned}
$$

A variational formulation

Consider the subvariety of X defined by fixed actions

$$M_a = \{S \in X : I_1(S) = a_1, \ldots I_m(S) = a_m\} \subseteq X$$

Variational principle: critical points of $\overline{H}(S)$ on M_a correspond to solutions of equation (10), with Lagrange multiplier Ω.

NB: All of $\overline{H}(S)$, $I_j(S)$ and M_a are invariant under the action of the torus $\mathbb{T}^m$; that is $\tau_\alpha : S(\xi) \mapsto S(\xi + \alpha)$, $\alpha \in \mathbb{T}^m$.

Two questions

- Two questions.
  - Do critical points exist on M_a?
    Note that the following operators are degenerate on the space of mappings X:

$$\Omega \cdot J^{-1}\partial_\xi S \;, \qquad \Omega \cdot J^{-1}\partial_\xi S - \delta_S^2 \overline{H}(0)$$

  - How to understand questions of multiplicity of solutions?
- Answers – in some cases:
  - Use the Nash – Moser method.
    Relies on solutions of the linearized equations, via resolvant expansions (Fröhlich – Spencer estimates)
  - Morse – Bott theory of critical $\mathbb{T}^m$ orbits.

The linearized wave equation

The nonlinear wave equation (5)

- Quadratic Hamiltonian,

$$H^{(2)}(q,p) = \sum_{k\in\Gamma'} \frac{1}{2}(\hat{p}_k^2 + \omega_k^2 \hat{q}_k^2) = \sum_{k\in\Gamma'} \omega_k I_k$$

- Fourier representation of torus mappings $S(\xi) : \mathbb{T}^m \mapsto \mathcal{H}$

$$S(x,\xi) = \sum_{k\in\Gamma'} S_k(\xi)\psi_k(x) = \sum_{k\in\Gamma', j\in\mathbb{Z}^m} S_{jk}\psi_k(x)e^{ij\cdot\xi}$$

The linearized operator

- The linearized equation can be rewritten

$$\Big(\delta_S^2\overline{H}^{(2)}(0) - \Omega\cdot\delta_S^2 I(0)\Big)S(x,\xi) = \sum_{j,k}\begin{pmatrix}\omega(k) & i\Omega\cdot j \\ -i\Omega\cdot j & \omega(k)\end{pmatrix}\begin{pmatrix}s_1(j,k)\\ s_2(j,k)\end{pmatrix}\psi_k(x)e^{ij\cdot\xi} \tag{11}$$

- Eigenvalues of this 2×2 block diagonal problem are

$$\mu(j,k) = \omega_k \pm \Omega\cdot j$$

Null space

- Choose $(\omega_{k_1}, \dots \omega_{k_m})$ linear frequencies, and a frequency vector $\Omega^0 = (\Omega_1^0, \dots \Omega_m^0)$ solving the resonance relations

$$\omega_{k_\ell} - \Omega^0 \cdot j_\ell = 0 \ .$$

- This identifies a null eigenspace in the space of mappings

$$X_1 \subseteq X \ .$$

Proposition

$X_1 \subseteq X$ is even dimensional; $\dim(X_1) = 2M \geq 2m$. It is possibly infinite dimensional

- Nonresonant case: $M = m$.
  Resonant case: $M > m$.

Lyapunov - Schmidt decomposition

- Decompose $X = \{S : \mathbb{T}^m \mapsto \mathcal{H}\} = X_1 \oplus X_2 = QX \oplus PX$.
- Equation (10) is equivalent to

$$Q\Big(J^{-1}\Omega \cdot \partial_\xi S \quad - \quad \text{grad}_v H(S)\Big) = 0\ , \tag{12}$$

$$P\Big(J^{-1}\Omega \cdot \partial_\xi S \quad - \quad \text{grad}_v H(S)\Big) = 0\ . \tag{13}$$

- Decompose the mappings $S = S_1 + S_2$ as well.
- Small divisor problem for $S_2 = S_2(S_1, \Omega)$, which one solves for $(S_1, \Omega) \in \mathcal{C}$ a Cantor set.

Reduced variational problem

It remains to solve the Q-equation (12). In case $M < +\infty$, it can be posed variationally (analogy with Weinstein - Moser theory).

- Define

$$\begin{aligned} I_j^1(S_1) &= I_j(S_1 + S_2(S_1, \Omega)) \\ \overline{H}^1(S_1) &= \overline{H}(S_1 + S_2(S_1, \Omega)) \\ M_a^1 &= \{S_1 \in X_1 \ : \ I_j^1(S_1) = a_j \ , \ j = 1 \ldots m\} \end{aligned}$$

- Critical points of $\overline{H}^1(S_1)$ on M_a^1 are solutions of (12) with action vector a.

Morse – Bott theory

The group $\mathbb{T}^m$ acts on M_a^1 leaving $\overline{H}^1(S_1)$ invariant. One seeks critical $\mathbb{T}^m$ orbits.

Question: How many critical orbits of $\overline{H}^1$ on M_a^1?

Depends upon its topology.

Conjecture

For given a there exist integers $p_1, \dots p_m$ such that $\sum_j p_j = M$ and

$$M_a^1 \simeq \otimes_{j=1}^m S^{2p_j - 1}$$

Morse – Bott theory

Check this fact, in endpoint cases.

- Periodic orbits $m = 1$, resonant case $M > 1$.

$$M_a^1 \simeq S^{2M-1} \ , \qquad M_a^1/\mathbb{T}^1 \simeq \mathbb{C}P_w(M-1)$$

This restates the estimate of Weinstein and Moser

$$\#\{\text{critical } \mathbb{T}^1 \text{ orbits}\} \geq M$$

Morse – Bott theory

- Nonresonant quasi-periodic orbits $m = M$.

$$M_a^1 \simeq \otimes_{j=1}^{M} S^1 \ , \qquad M_a^1/\mathbb{T}^m \simeq \text{a point}$$

  This corresponds to a KAM torus.

- The case $m = 2 \leq M$ occurs in the problem of doubly periodic traveling wave patterns on the surface of water.

$$M_a^1 \simeq S^{2p-1} \otimes S^{2(M-p)-1}$$

Doubly periodic traveling waves of hexagonal form

topology of links

Theorem

(Chaperon, Bosio & Meersmann (2006)) The topology of links M_a^1 *can be complex. There are cases in which*

$$M_a^1 \simeq S^{2p_1-1} \# S^{2p_2-1} \cdots \otimes S^{2p_\ell - 1}$$

Furthermore, there are more complex quantities than this.

Proof: combinatorics and cohomolological calculations.

Conjecture

The number of distinct critical $\mathbb{T}^m$ *orbits of* $\overline{H}^1$ *on* M_a^1 *is bounded below:*

$$\#\{\text{critical orbits of } \overline{H}^1 \text{ on } M_a^1\} \geq (M - m + 1) \ .$$

methods of KAM theory

- Multiple approaches to KAM theory and PDE
  1. Classical methods of iterations of canonical transformations
  2. Convergence of Lindstedt series, and cancellations
  3. Nash – Moser, inverse of the linearized operator by resolvant expansions (Fröhlich – Spencer estimates)
- History:
  - Periodic solutions:
    Lyapunov (1907),
    A. Weinstein (1973), Moser (1976)
  - Quasiperiodic solutions:
    Kolmogorov, Arnold & Moser (1954)(1961)(1962)
    $\sim$ Melnikov (1968)

recent advances in KAM theory

- finite dimensions
  Eliasson, Pöschel, Kuksin, Gallavotti et al, de la Llave, Wayne, Bourgain, J. You, C.Q. Cheng, . . .

- Partial differential equations:
  Kuksin, Wayne, W. C., Bourgain, Chierchia, Falcolini, Pöschel, Eliasson, Su, Grébert, You, Kappeler, Bambusi, Plotnikov, Toland, Iooss, Berti, Bolle, Yi, Yuan, Geng . . .

Resolvant expansions

Linearize (13) about an approximate embedding of an invariant torus

$$S^0 = S_1 + S_2^0 \; .$$

- The linearized equation

$$P\big(\delta^2_{S_2}\overline{H}(S^0) - \Omega \cdot \delta^2_{S_2} I(S^0)\big)PV = G \; , \tag{14}$$

- In eigenfunction expansion,

$$\begin{aligned} &P\big(\delta^2_{S_2}\overline{H}(S_1 + S_2^0) - \Omega \cdot \delta^2_{S_2} I(S_1 + S_2^0)\big)PV \\ &= P\left(\mathrm{diag}_{2\times 2}\begin{pmatrix} \omega(k) & i\Omega \cdot j \\ -i\Omega \cdot j & \omega(k) \end{pmatrix} + W((j,k),(j',k'))\right)PV \\ &= G \; . \end{aligned}$$

for lattice sites $y = (j,k), y' = (j',k') \in \mathbb{Z}^m \oplus \Gamma'$

Resolvant expansions

Definition

A lattice site $y = (j, k) \in \mathbb{Z}^m \oplus \Gamma'$ is d_0-*singular* for Ω when

$$|\omega(k) \pm \Omega \cdot j| < d_0 \ ,$$

and *regular* otherwise.

Theorem

For $A \subseteq \mathbb{Z}^m \oplus \Gamma'$ *having only regular sites, and for* $|W|_{Op} < d_0/2$, *then*

$$|\big(P(\delta^2_{S_2}\overline{H}(S_1 + S^0_2) - \Omega \cdot \delta^2_{S_2} I(S_1 + S^0_2))P\big)^{-1}_A|_{Op(A)} \leq \frac{4}{d_0} \ .$$

Fröhlich – Spencer estimates

- Fröhlich – Spencer estimates are used to add sets S of singular sites $y = (j,k) \notin A \subseteq \mathbb{Z}^m \oplus \Gamma'$ to the operator inverse.
- Estimates depend upon two properties of the operator

$$D(\Omega) + W := P\Big(\delta^2_{S_2}\overline{H}(S_1 + S^0_2) - \Omega \cdot \delta^2_{S_2} I(S_1 + S^0_2)\Big)P \ .$$

To explain this:

Let $H_B := (D(\Omega) + W)|_B$ for subsets $B \subseteq \mathbb{Z}^m \oplus \Gamma'$

Let $R_n \to \infty$ be a sequence used to control convergence.

Fröhlich – Spencer estimates

- **nonresonance.** If $y = (j,k)$ and $y' = (j',k')$ in $\mathbb{Z}^m \oplus \Gamma'$ satisfy $R_n < |y|, |y'| \le R_{n+1}$, then

$$\begin{aligned} d_n &< |\omega(k) - \Omega \cdot j| < d_0 \\ d_n &< |\omega(k') - \Omega \cdot j'| < d_0 \ . \end{aligned}$$

- **separation.** Suppose that two singular sites y, y' satisfy $R_n < |y|, |y'| \le R_{n+1}$.
  then either;
  - $\mathrm{dist}(y,y') < R_n^\alpha$ and they are included in the same singular set S,

    or else
  - $\mathrm{dist}(y,y') \gg R_n^\gamma$

  for appropriate $0 < \alpha \ll 1$, $0 \ll \gamma$.

Geometry of the singular sites

Figure: Wavenumber/frequency lattice and singular sites S_n

Resolvant expansions

- Block diagonal decomposition of the Hamiltonian

$$H_B = H_A \oplus_j H_{S_j} + \Gamma \ ,$$

- Inverting H_B the generalized resolvant identity is that

$$G_B = G_A \oplus_j G_{S_j} + G_A \oplus_j G_{S_j} \Gamma G_B \ ,$$

- Iterate to arrive at the expression

$$G_B = G_A \oplus_j G_{S_j} + \sum_{m=1}^{\infty} G_A \oplus_j G_{S_j} (\Gamma G_A \oplus_j G_{S_j})^m \ .$$

Estimate the convergence of this expression using the spacing of the singular sites

Thank you