Invariant tori for Hamiltonian PDE

Walter Craig

Department of Mathematics & Statistics

Peking University, Beijing – August 13, 2007
Hamiltonian PDE

- Flow in *phase space*, where \( v \in \mathcal{H} \) a Hilbert space

\[
\partial_t v = J \text{grad}_v H(v) , \quad v(x, 0) = v^0(x) ,
\]  

(1)

- Symplectic form

\[
\omega(X, Y) = \langle X, J^{-1}Y \rangle_{\mathcal{H}} , \quad J^T = -J .
\]

- The flow \( v(x, t) = \varphi_t(v^0(x)) \)

- Interested in orbits where

\[
\{ \varphi_t(v^0) : t \in \mathbb{R} \} = \mathbb{T}^m
\]

an \( m \)-dimensional torus. This gives stable motions of (1).
Outline

1. Hamiltonian PDE - examples
2. A variational formulation for invariant tori
3. The linearized operator
4. Equivariant Morse theory
5. Estimates for small divisors
Hamiltonian systems

- A finite dimensional Hamiltonian system, \( H(q,p) : \mathbb{R}^{2n} \rightarrow \mathbb{R} \)

\[
\begin{align*}
\dot{q} &= \text{grad}_p H, \\
\dot{p} &= -\text{grad}_q H \tag{2}
\end{align*}
\]

- Ask that \( \text{grad} \, H(0) = 0 \) and \( \text{hess} \, H(0) > 0 \),

\[
H = H^{(2)} + R
\]

After a change of variables,

\[
H^{(2)} = \frac{1}{2} |p|^2 + \frac{1}{2} \langle q, Aq \rangle
\]

\[
A = \text{diag}_{k=1\ldots n} (\omega_k^2)
\]
The harmonic oscillator

- **Linearized problem** about \((q, p) = 0\)

\[
\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_q H(2) \\ \partial_p H(2) \end{pmatrix}
\] (3)

- The linear flow, setting \(\xi_k(t) = t\omega_k\)

\[
\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \Phi_t \begin{pmatrix} q^0 \\ p^0 \end{pmatrix} = \text{diag}_{2\times2} \begin{pmatrix} \cos(\xi_k(t)) & \sin(\xi_k(t))/\omega_k \\ -\omega_k \sin(\xi_k(t)) & \cos(\xi_k(t)) \end{pmatrix} \begin{pmatrix} q^0 \\ p^0 \end{pmatrix}
\]

- Solutions lie on **tori** of dimension \(m = \dim_{\mathbb{Q}} \{\omega_1, \ldots, \omega_n\}\).

\[\mathbb{T}^m = \left\{ \Phi_t(q^0, p^0) : t \in \mathbb{R} \right\}\]
The nonlinear wave equation

- On a domain $\mathbb{T}^d = \mathbb{R}^d / \Gamma$, for period lattice $\Gamma$

$$\partial_t^2 u - \Delta u + g(x, u) = 0 .$$  

(4)

(Alternatively, $u = 0$ on the boundary of a domain $D \subseteq \mathbb{R}^d$).

- The Energy is

$$H(u, p) = \int_{\mathbb{T}^d} \frac{1}{2} p^2 + \frac{1}{2} |\nabla u|^2 + G(x, u) \, dx ,$$

- Equation (4) can be rewritten as

$$\partial_t u = \text{grad}_p H(u, p) = p$$

$$\partial_t p = -\text{grad}_u H(u, p) = \Delta u - \partial_u G(x, u) ,$$

in Darboux coordinates, where $g(x, \cdot) = \partial_u G(x, \cdot)$.
Linearized wave equation

- Suppose the Taylor series for $G(x, u)$ is
  \[
  G(x, u) = \frac{1}{2} g_1(x) u^2 + \frac{1}{3} g_2(x) u^3 + \ldots
  \]

- Then the Hamiltonian takes the form $H = H^{(2)} + R$, with
  \[
  H^{(2)} = \int_{\mathbb{T}^d} \frac{1}{2} p^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} g_1(x) u^2 \, dx
  \]
  \[
  = \sum_{k \in \Gamma'} \frac{1}{2} |\hat{p}_k|^2 + \frac{1}{2} \omega_k^2 |\hat{u}_k|^2
  \]

- Eigenfunction/eigenvalue pairs $(\psi_k(x), \omega_k^2)$ for the operator
  \[
  L(g_1) \psi_k = (-\Delta + g_1(x)) \psi_k = \omega_k^2 \psi_k
  \]
The linearized flow

- Solutions of the linear wave equations are

\[
\begin{pmatrix}
  u(x, t) \\
  p(x, t)
\end{pmatrix} = \Phi_t \begin{pmatrix}
  u^0(x) \\
  p^0(x)
\end{pmatrix}
\]

\[
= \sum_{k \in \Gamma'} \psi_k(x) \begin{pmatrix}
  \cos(\xi_k(t)) & \sin(\xi_k(t))/\omega_k \\
  -\omega_k \sin(\xi_k(t)) & \cos(\xi_k(t))
\end{pmatrix} \begin{pmatrix}
  \hat{u}_k^0 \\
  \hat{p}_k^0
\end{pmatrix}
\]

- This is the harmonic oscillator with frequencies \( \{\omega_k\}_{k \in \Gamma'} \).
- Generically, \( \dim_{\mathbb{Q}} \{\omega_1, \ldots\} = \infty \).
Basic facts

Some basic facts about the flow of the linearized problem:

- The Hamiltonian is preserved by the flow:

\[ H^{(2)}(\Phi_t(v)) = H^{(2)}(v) \]

Conservation of energy

- All of the actions are preserved:

\[ I_k(v) := \frac{1}{2}(\omega_k|\hat{u}_k|^2 + \frac{1}{\omega_k}|\hat{p}_k|^2) = I_k(\Phi_t(v)) \]

The moment map: \((\hat{u}, \hat{p}) \mapsto I\)

- The phases evolve linearly in time; \(t \mapsto \{\xi_k(t) = \omega_k t\}_{k \in \Gamma'}\).

Solutions are periodic when \(\dim_{\mathbb{Q}} \{\omega_{j_1}, \omega_{j_2}, \ldots\} = 1\), quasi-periodic when \(< +\infty\) and otherwise almost-periodic.
Basic questions

Basic questions regarding the flow of the nonlinear systems

\[ \partial_t v = J \text{grad}_v \left( H^{(2)} + R \right) . \]  \hspace{1cm} (5)

(1) Whether \textit{some} solutions are

- periodic \( \mathbb{T}^1 \hookrightarrow \mathcal{H} \)
- quasi-periodic \( \mathbb{T}^m \hookrightarrow \mathcal{H} \), \( m < +\infty \)
- almost-periodic (and even Lagrangian invariant tori).

Corresponding to \textit{stable motions} of (5).
Basic questions

(2) Given data $v^0 \in \mathcal{H}$, whether

- The flow $\varphi_t(v^0)$ remains in $\mathcal{H}$ for all time (global well-posedness of the PDE),
- for $v^0 \in B_\varepsilon(0)$ then $\varphi_t(v^0) \in B_\delta(0)$ for all $t \in \mathbb{R}$ (stability),
- action variables change by controlled amounts

$$|I_k(\varphi_t(v)) - I_k(v)| < \varepsilon^\alpha$$

for $|t| < T(\varepsilon) \sim \exp 1/\varepsilon^\beta$ (Nekhoroshev stability).
Basic questions

(3) Whether for some solutions there are lower bounds on the growth of the action variables $I_k(\varphi_t(v^0)) - I_k(v^0)$, or of Sobolev norms for large $|t| \gg 1$

$$\|\varphi_t(v^0)\|_{H^s} \geq \delta(t), \quad s \gg 1 \quad (\text{Arnold diffusion}).$$
Further examples

- **Nonlinear Schrödinger equation**

  \[ i\partial_t u - \frac{1}{2}\Delta_x u + Q(x, u, \overline{u}) = 0 , \quad x \in \mathbb{T}^d \]  
  
  with Hamiltonian

  \[ H_{NLS}(u) = \int_{\mathbb{T}^d} \frac{1}{2}|\nabla u|^2 + G(x, u, \overline{u}) \, dx , \quad \partial_{\overline{u}} G = Q . \]

  Rewritten

  \[ \partial_t u = i \text{grad}_{\overline{u}} H_{NLS} \]
**Further examples**

- **Korteweg – de Vries equation**

  \[ \partial_t q = \frac{1}{6} \partial_x^3 q - \partial_x (\partial_q G(x, q)) \quad , \quad x \in \mathbb{T}^1 \]  
  
  The Hamiltonian is

  \[ H_{KdV}(q) = \int_{\mathbb{T}^1} \frac{1}{12} (\partial_x q)^2 + G(x, q) \, dx \]  
  
  Rewritten

  \[ \partial_t q = J \, \text{grad}_q H_{KdV} \quad , \quad \text{where} \quad J = -\partial_x \]
Further examples

- **Surface water waves:** The fluid domain

\[ \Sigma(\eta) = \{ y = \eta(x, t) : x \in \mathbb{T}^{d-1} \} \]

Velocity field, a potential flow \( \mathbf{u} = \nabla \varphi , \quad \Delta \varphi = 0. \)

**Hamilton’s principle**

\[
\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \text{grad}_\eta H(\eta, \xi) \\ \text{grad}_\xi H(\eta, \xi) \end{pmatrix}, \tag{8}
\]

Canonical conjugate variables:

- \( \eta(x) \) (free surface elevation)
- \( \xi(x) = \varphi(x, \eta(x)) \)

(boundary values of the velocity potential)
The Hamiltonian is

\[ H(\eta, \xi) = \int \frac{1}{2} \xi G(\eta) \xi + \frac{1}{2} g\eta^2 \, dx, \]

where \( G(\eta) \) is the Dirichlet – Neumann operator on \( \Sigma(\eta) \).

Taylor series of \( G(\eta) = \sum_{j \geq 0} G^{(j)}(\eta) \)

\[
G^{(0)} = D \tanh(hD) , \quad D := -i \partial_x \\
G^{(1)}(\eta) = D\eta(x)D - G^{(0)}\eta(x)G^{(0)}
\]

Hadamard’s variational principle
Interactions of solitary waves

- Interaction of a head-on collision of two solitary waves of amplitudes $S/h = 0.4$
  
  (a) (b)

- Long time behavior after collision
An invariant torus

- Mapping of a torus \( S(\xi) : \mathbb{T}^m \mapsto \mathcal{H} \)
- Flow invariance \( S(\xi + t\Omega) = \varphi_t(S(\xi)) \)
  Frequency vector \( \Omega \in \mathbb{R}^m \).
- This implies that both

\[
\partial_t S = J \text{ grad}_v H(S), \quad \text{and} \quad \partial_t S = \Omega \cdot \partial_\xi S. \quad (9)
\]

- **Problem:** Solve (9) for \((S(\xi), \Omega)\).
  This is generally a small divisor problem.

  Rewrite (9) as

\[
J^{-1} \Omega \cdot \partial_\xi S - \text{grad}_v H(S) = 0. \quad (10)
\]
Consider the space of mappings $S \in X := \{S(\xi) : \mathbb{T}^m \leftrightarrow \mathcal{H}\}$.

- Define action functionals
  \[
  I_j(S) = \frac{1}{2} \int_{\mathbb{T}^m} \langle S, J^{-1} \partial_{\xi_j} S \rangle \, d\xi
  \]
  \[
  \delta_S I_j = J^{-1} \partial_{\xi_j} S
  \]

- The moment map for mappings

- The average Hamiltonian
  \[
  \overline{H}(S) = \int_{\mathbb{T}^m} H(S(\xi)) \, d\xi
  \]
  \[
  \delta_S \overline{H} = \text{grad}_v H(S)
  \]
A variational formulation

Consider the subvariety of $X$ defined by fixed actions

$$M_a = \{ S \in X : I_1(S) = a_1, \ldots, I_m(S) = a_m \} \subseteq X$$

**Variational principle:** critical points of $\overline{H}(S)$ on $M_a$ correspond to solutions of equation (10), with Lagrange multiplier $\Omega$.

**NB:** All of $\overline{H}(S)$, $I_j(S)$ and $M_a$ are invariant under the action of the torus $\mathbb{T}^m$; that is $\tau_\alpha : S(\xi) \mapsto S(\xi + \alpha)$, $\alpha \in \mathbb{T}^m$. 
Two questions

- Two questions.
  - Do critical points exist on $M_a$?
    Note that the following operators are degenerate on the space of mappings $X$:
    $$\Omega \cdot J^{-1} \partial_\xi S, \quad \Omega \cdot J^{-1} \partial_\xi S - \delta_3^2 \overline{\mathcal{H}}(0)$$

- How to understand questions of multiplicity of solutions?

- Answers – in some cases:
  - Use the Nash – Moser method.
    Relies on solutions of the linearized equations, via resolvant expansions (Fröhlich – Spencer estimates)
  - Morse – Bott theory of critical $\mathbb{T}^m$ orbits.
The linearized wave equation

The nonlinear wave equation (5)

- Quadratic Hamiltonian,

\[ H^{(2)}(q, p) = \sum_{k \in \Gamma'} \frac{1}{2} (\hat{p}_k^2 + \omega_k^2 \hat{q}_k^2) = \sum_{k \in \Gamma'} \omega_k I_k \]

- Fourier representation of torus mappings \( S(\xi) : \mathbb{T}^m \mapsto \mathcal{H} \)

\[ S(x, \xi) = \sum_{k \in \Gamma'} S_k(\xi) \psi_k(x) = \sum_{k \in \Gamma', j \in \mathbb{Z}^m} S_{jk} \psi_k(x) e^{ij \cdot \xi} \]
The linearized operator

- The linearized equation can be rewritten

\[
\left( \delta_S^2 H^{(2)}(0) - \Omega \cdot \delta_S^2 I(0) \right) S(x, \xi) = \sum_{j,k} \begin{pmatrix} \omega(k) & i\Omega \cdot j \\ -i\Omega \cdot j & \omega(k) \end{pmatrix} \begin{pmatrix} s_1(j,k) \\ s_2(j,k) \end{pmatrix} \psi_k(x) e^{ij \cdot \xi}
\]

- Eigenvalues of this $2 \times 2$ block diagonal problem are

\[
\mu(j, k) = \omega_k \pm \Omega \cdot j
\]
Choose \((\omega_{k_1}, \ldots, \omega_{k_m})\) linear frequencies, and a frequency vector \(\Omega^0 = (\Omega^0_1, \ldots, \Omega^0_m)\) solving the resonance relations

\[
\omega_{k_\ell} - \Omega^0 \cdot j_\ell = 0.
\]

This identifies a null eigenspace in the space of mappings \(X_1 \subseteq X\).

**Proposition**

\(X_1 \subseteq X\) is even dimensional; \(\dim(X_1) = 2M \geq 2m\). It is possibly infinite dimensional

- Nonresonant case: \(M = m\).
- Resonant case: \(M > m\).
Lyapunov - Schmidt decomposition

- Decompose $X = \{ S : \mathbb{T}^m \hookrightarrow \mathcal{H} \} = X_1 \oplus X_2 = QX \oplus PX$.

- Equation (10) is equivalent to
  \begin{align}
  Q(J^{-1}\Omega \cdot \partial_S - \text{grad}_vH(S)) &= 0, \\
  P(J^{-1}\Omega \cdot \partial_S - \text{grad}_vH(S)) &= 0. 
  \end{align}

- Decompose the mappings $S = S_1 + S_2$ as well.

- Small divisor problem for $S_2 = S_2(S_1, \Omega)$, which one solves for $(S_1, \Omega) \in \mathcal{C}$ a Cantor set.
It remains to solve the Q-equation (12). In case $M < +\infty$, it can be posed variationally (analogy with Weinstein - Moser theory).

- Define

\[
\begin{align*}
I^1_j(S_1) &= I_j(S_1 + S_2(S_1, \Omega)) \\
\overline{H}^1(S_1) &= \overline{H}(S_1 + S_2(S_1, \Omega)) \\
M^1_a &= \{ S_1 \in X_1 : I^1_j(S_1) = a_j, \ j = 1 \ldots m \}
\end{align*}
\]

- Critical points of $\overline{H}^1(S_1)$ on $M^1_a$ are solutions of (12) with action vector $a$. 
Morse – Bott theory

The group $\mathbb{T}^m$ acts on $M^1_a$ leaving $\overline{H}^1(S_1)$ invariant. One seeks critical $\mathbb{T}^m$ orbits.

Question: How many critical orbits of $\overline{H}^1$ on $M^1_a$?

Depends upon its topology.

Conjecture

For given $a$ there exist integers $p_1, \ldots, p_m$ such that $\sum_j p_j = M$ and

$$M^1_a \sim \bigotimes_{j=1}^m S^{2p_j-1}$$
Morse – Bott theory

Check this fact, in endpoint cases.

- Periodic orbits $m = 1$, resonant case $M > 1$.

$$M^1_a \simeq S^{2M-1}, \quad M^1_a/\mathbb{T}^1 \simeq \mathbb{C}P_w(M-1)$$

This restates the estimate of Weinstein and Moser

$$\# \{ \text{critical } \mathbb{T}^1 \text{ orbits} \} \geq M$$
Morse – Bott theory

- **Nonresonant quasi-periodic orbits** $m = M$.

\[ M_a^1 \cong \bigotimes_{j=1}^M S^1, \quad M_a^1 / \mathbb{T}^m \cong \text{a point} \]

This corresponds to a KAM torus.

- The case $m = 2 \leq M$ occurs in the problem of doubly periodic traveling wave patterns on the surface of water.

\[ M_a^1 \cong S^{2p-1} \otimes S^{2(M-p)-1} \]
Doubly periodic traveling waves of hexagonal form
Theorem

(Chaperon, Bosio & Meersmann (2006)) The topology of links $M^1_a$ can be complex. There are cases in which

$$M^1_a \cong S^{2p_1-1} \# S^{2p_2-1} \cdots \otimes S^{2p_{\ell}-1}$$

Furthermore, there are more complex quantities than this.

Proof: combinatorics and cohomolological calculations.

Conjecture

The number of distinct critical $\mathbb{T}^m$ orbits of $\overline{H}^1$ on $M^1_a$ is bounded below:

$$\# \{ \text{critical orbits of } \overline{H}^1 \text{ on } M^1_a \} \geq (M - m + 1).$$
Multiple approaches to KAM theory and PDE

1. Classical methods of iterations of canonical transformations
2. Convergence of Lindstedt series, and cancellations
3. Nash – Moser, inverse of the linearized operator by resolvent expansions (Fröhlich – Spencer estimates)

History:

- Periodic solutions:
  Lyapunov (1907),
  A. Weinstein (1973), Moser (1976)
- Quasiperiodic solutions:
  ~ Melnikov (1968)
recent advances in KAM theory

- finite dimensions
  Eliasson, Pöschel, Kuksin, Gallavotti et al, de la Llave, Wayne, Bourgain, J. You, C.Q. Cheng, ...

- Partial differential equations:
  Kuksin, Wayne, W. C., Bourgain, Chierchia, Falcolini, Pöschel, Eliasson, Su, Grébert, You, Kappeler, Bambusi, Plotnikov, Toland, Iooss, Berti, Bolle, Yi, Yuan, Geng ...
Resolvant expansions

Linearize (13) about an approximate embedding of an invariant torus

\[ S^0 = S_1 + S_2^0. \]

- The linearized equation

\[
P \left( \delta_{S_2}^2 \mathcal{H}(S^0) - \Omega \cdot \delta_{S_2}^2 I(S^0) \right) PV = G, \tag{14}
\]

- In eigenfunction expansion,

\[
P \left( \delta_{S_2}^2 \mathcal{H}(S_1 + S_2^0) - \Omega \cdot \delta_{S_2}^2 I(S_1 + S_2^0) \right) PV
\]

\[
= P \left( \text{diag}_{2 \times 2} \begin{pmatrix} \omega(k) & i\Omega \cdot j \\ -i\Omega \cdot j & \omega(k) \end{pmatrix} + W((j, k), (j', k')) \right) PV
\]

\[
= G.
\]

for lattice sites \( y = (j, k), y' = (j', k') \in \mathbb{Z}^m \oplus \Gamma' \).
Definition

A lattice site \( y = (j, k) \in \mathbb{Z}^m \oplus \Gamma' \) is \( d_0 \)-singular for \( \Omega \) when

\[
|\omega(k) \pm \Omega \cdot j| < d_0,
\]

and regular otherwise.

Theorem

For \( A \subseteq \mathbb{Z}^m \oplus \Gamma' \) having only regular sites, and for \( |W|_{Op} < d_0/2 \), then

\[
\left| \left( P(\delta_{S_2}^2 \overline{H}(S_1 + S_0^0) - \Omega \cdot \delta_{S_2}^2 I(S_1 + S_2^0))P \right)^{-1} \right|_{Op(A)} \leq \frac{4}{d_0}.
\]
Fröhlich – Spencer estimates

- Fröhlich – Spencer estimates are used to add sets $S$ of singular sites $y = (j, k) \notin A \subseteq \mathbb{Z}^m \oplus \Gamma'$ to the operator inverse.

- Estimates depend upon two properties of the operator

$$D(\Omega) + W := P\left(\delta_{S_2}^2 \overline{H}(S_1 + S_2^0) - \Omega \cdot \delta_{S_2}^2 I(S_1 + S_2^0)\right)P.$$ 

To explain this:

Let $H_B := (D(\Omega) + W)|_B$ for subsets $B \subseteq \mathbb{Z}^m \oplus \Gamma'$

Let $R_n \to \infty$ be a sequence used to control convergence.
Fröhlich – Spencer estimates

- **nonresonance.** If \( y = (j, k) \) and \( y' = (j', k') \) in \( \mathbb{Z}^m \oplus \Gamma' \) satisfy \( R_n < |y|, |y'| \leq R_{n+1} \), then

\[
\begin{align*}
  d_n &< |\omega(k) - \Omega \cdot j| < d_0 \\
  d_n &< |\omega(k') - \Omega \cdot j'| < d_0.
\end{align*}
\]

- **separation.** Suppose that two singular sites \( y, y' \) satisfy \( R_n < |y|, |y'| \leq R_{n+1} \).

then either;

- \( \text{dist}(y, y') < R_n^\alpha \) and they are included in the same singular set \( S \),
- or else
  - \( \text{dist}(y, y') \gg R_n^\gamma \)

for appropriate \( 0 < \alpha \ll 1, 0 \ll \gamma \).
Geometry of the singular sites

Figure: Wavenumber/frequency lattice and singular sites $S_n$
Resolvant expansions

- Block diagonal decomposition of the Hamiltonian
  \[ H_B = H_A \oplus_j H_{S_j} + \Gamma , \]

- Inverting \( H_B \) the generalized resolvant identity is that
  \[ G_B = G_A \oplus_j G_{S_j} + G_A \oplus_j G_{S_j} \Gamma G_B , \]

- Iterate to arrive at the expression
  \[ G_B = G_A \oplus_j G_{S_j} + \sum_{m=1}^{\infty} G_A \oplus_j G_{S_j} (\Gamma G_A \oplus_j G_{S_j})^m . \]

Estimate the convergence of this expression using the spacing of the singular sites.
Thank you