Math 742. Semester 2, 2015-2016
Problem Set #2

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Problem 1. (Symmetries of solutions of the wave equation)
Consider solutions to the initial value problem for the wave equation

$$\begin{aligned} &\partial_t^2 u - \Delta u = 0 \ , \qquad x \in \mathbb{R}^n \\ &u(0,x) = f(x) \ , \quad \partial_t u(0,x) = g(x) \qquad \text{initial data} \end{aligned}$$

If $f(x) = 0$, show that $u(t,x)$ is odd under time reversal $t \mapsto -t$.

If $g(x) = 0$ show that $u(t,x)$ is even under time reversal $t \mapsto -t$.

What conditions would you ask of the data $f(x)$ and $g(x)$ so that the solution is odd in the transformation $x \mapsto -x$. Or so that the solution is even in $x \mapsto -x$. What condition would ensure that the solution is radially symmetric in x (*ie* a function of $|x| = r$ alone).

Problem 2. This problem concerns the Cauchy problem for the wave equation

$$\partial_t^2 u - \Delta u = 0 \ , \qquad x \in \mathbb{R}^n$$

for *Cauchy data* (f,g) posed on a space-like Cauchy hypersurface

$$\Gamma = \{(t,x) : t = v x_1\} \ , \qquad |v| < 1 \ .$$

Show that the problem can be reduced to the standard initial value problem posed on the initial hypersurface $\{(t',x') : t' = 0\}$ in a new coordinate system given through the Lorentz transformation

$$x_1' = \frac{x - vt}{\sqrt{1 - |v|^2}} \ , \quad x_2' = x_2 \ldots x_n' = x_n \ , \quad t' = \frac{t - v x_1}{\sqrt{1 - |v|^2}} \ .$$

In cases $n = 1$ and $n = 3$ give an explicit expression in the original (t,x) coordinates for the solution to the Cauchy problem described in this problem.

Problem 3. (initial - boundary value problems for the wave equation)
(a) Solve the initial-boundary value problem for the wave equation in $x \in \mathbb{R}^1_+$, $t \in \mathbb{R}^1_+$ consisting of

$$\begin{aligned}
&\partial_t^2 u - \partial_x^2 u = 0 \ , \\
&u(0,x) = f(x) \ , \quad \partial_t u(0,x) = g(x) \ , \\
&\partial_x u(t,0) - \alpha u(t,0) = 0 \ ,
\end{aligned}$$

where the latter boundary condition is imposed on the fixed boundary $\{x = 0\}$. These are known as *impedance* boundary conditions.
Hint: Use the expression for $u(t,x) = V(x+t) + W(x-t)$ and the method of images to select an appropriate choice of the values of $W(x)$ for $x < 0$.
(b) For which range of values of α does the energy of the solution decrease in time; here the energy is given by

$$E(u) = \tfrac{1}{2} \int_0^{+\infty} (\partial_t u(t,x))^2 + (\partial_x u(t,x))^2 \, dx \ .$$

Problem 4. Using the Fourier transform, describe solutions of the initial value problem for the Klein - Gordon equation

$$\begin{aligned}
&\partial_t^2 u - \partial_x^2 u + m^2 u = 0 \ , \\
&u(0,x) = f(x) \ , \quad \partial_t u(0,x) = g(x) \ .
\end{aligned}$$

Derive a conserved energy integral quantity for solutions.