Math 742. Semester 2, 2015-2016
Problem Set #3

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Problem 1. (Maxwell's equations)
Maxwell's equations for electromagnetic fields in three space dimensions $x \in \mathbb{R}^3$ can be stated in terms of the electric field $E(t,x)$ and the magnetic field $B(t,x)$ as follows:

$$\partial_t E = \nabla \times B - J \ , \qquad \partial_t B = -\nabla \times E \tag{1}$$

$$\nabla \cdot E = \rho \ , \qquad \nabla \cdot B = 0 \ . \tag{2}$$

The quantities $\rho(t,x)$ and $J(t,x)$ are respectively the charge density and the current density vector, and are given as input to the problem. As is implied by the notation, ρ is a scalar function while E, B and J are three dimensional vectors whose entries depend on (t,x).
(a) Prove the vector calculus identity that for any three dimensional vector $F(x) \in C^2$ then

$$\nabla \cdot (\nabla \times F) = 0 \ .$$

(b) Show that Maxwell's equations (1) can be written in symmetric hyperbolic form. Note that the pair of equations (2) are constraints.
(c) Show that under the conditions that $\partial_t \rho = 0$ and $\nabla \cdot J = 0$, the constraint equations are conserved by solutions. That is, if initially $\nabla \cdot E(0,x) = \rho(x)$ and $\nabla \cdot B(0,x) = 0$ then this condition continues to hold for $t \in \mathbb{R}$.

Problem 2. (Gårding's condition of hyperbolicity for systems)
The condition of Gårding for well posedness of a constant coefficient system of equations is similar to that of a scalar equation. Namely, consider the canonical problem

$$P(D,\tau)u = 0 \ , \quad t > 0$$
$$\tau^k u(0,x) = 0 \ , k = 0, \dots m-2 \ , \quad \tau^{m-1} u(0,x) = g(x) \in \mathcal{S} \ .$$

The matrix $P(\xi,\lambda)$, which depends upon $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ is the symbol of the system of equations. We assume that the matrix $P(0,1)$ is invertible, so that without loss of generality $P(0,1) = I$. This implies that the initial hypersurface $\{(t,x) : t = 0\}$ is noncharacteristic.
(a) Show that the general solution to the initial value problem can be given in terms of the solution map $u(t,x) = S(t)g(x)$ for the canonical problem.
(b) Gårding's condition of hyperbolicity for systems is that the roots λ of the polynomial $q(\xi,\lambda) = \det P(\xi,\lambda) = 0$ satisfy the condition

$$\operatorname{im} \lambda(\xi) \geq -C \ .$$

Construct the solution map $S(t)$ using complex variables techniques, and prove the property that $S(t) : \mathcal{S} \to \mathcal{S}$, namely Schwartz class initial data leads to Schwartz class solutions.

(c) Show that a linear symmetric hyperbolic system with constant coefficient matricies A_0, A_j, $j = 1, \dots n$ and B satisfies the Gårding condition of hyperbolicity.

Problem 3. (Sobolev embedding theorem)
This problem has to do with inequalities between norms in spaces of functions.

(a) Prove the Sobolev embedding theorem, that for s a real parameter such that $s > n/2$ then

$$|u(x)|_{L^\infty} \le C_{ns} \|u\|_{H^s} ,$$

where the Sobolev norm $\|u\|_{H^s}$ is defined by

$$\|u\|_{H^s}^2 := \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi .$$

(b) Show the further refinement that if $n/2 < s \le n/2 + 1$ then every function $u \in H^s$ is also Hölder continuous, $u \in C^\alpha(\mathbb{R}^n)$, for any Hölder exponent $0 \le \alpha < s - n/2$. Therefore if $u \in H^r$ for some $r > n/2$ then in fact $u \in C^{k,\alpha}$ for $k = [r - n/2]$ and $0 \le \alpha < r - n/2 - k$.

Problem 4. (quantum harmonic oscillator)
Solving the Schrödinger equation for the quantum harmonic oscillator

$$\frac{1}{i}\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi + \frac{1}{2}x^2 \psi ,$$

by separation of variables, one is led to the expression

$$\psi(t, x) = \sum_k a_k e^{i\lambda_k x} \varphi_k(x) .$$

The eigenfunctions and eigenvalue pairs $(\varphi_k(x), \lambda_k)$ are given by the problem in spectral theory;

$$-\frac{1}{2}\partial_x^2 \varphi_k(x) + \frac{1}{2}x^2 \varphi_k(x) = \lambda_k \varphi_k(x) . \tag{3}$$

(a) Show that under Fourier transform the eigenvalue problem (3) is transformed to itself. Conclude that the eigenfunctions of the quantum harmonic oscillator are also eigenfunctions of the Fourier transform, and vice versa.

(b) Derive the eigenfunctions for (3), and show that the corresponding eigenvalues λ_k are all real. What are they.

(c) What are the corresponding eigenvalues of the Fourier transform?