Math 742. Semester 2, 2015-2016
Problem Set #4

Instructor: Walter Craig
Problem set due date: Thursday March 24, 2016

Problem 1. (maximum principle #1) Consider the elliptic partial differential operator

$$Lu = \sum_{j,k=1}^{n} a_{jk}(x)\partial_{x_j}\partial_{x_k}u + \sum_{j=1}^{k} b_j(x)\partial_{x_j}u + cu \ .$$

(a) Assume that $u \in C^2(\bar{D})$ is a subsolution of L, meaning that $Lu \geq 0$. Show that if $c < 0$ and if u has a nonnegative maximum $u(x_0) = M := \max_{\bar{D}}(u(x))$ at some interior point $x_0 \in D$, then necessarily $u(x) = M$ and furthermore $M = 0$.

(b) Again assume that $u \in C^2(\bar{D})$ and $c \leq 0$, and now let $Lu = 0$. Show that if $u(x) = 0$ on ∂D then $u = 0$.

(c) Give an example of an elliptic operator L such that $c > 0$, and such that there exists $u \in C^2(\bar{D})$ such that $Lu = 0$ and $u(x) = 0$ for $x \in \partial D$, where $u(x)$ is not identically zero.

Problem 2. (maximum principle #2) Again consider the elliptic operator L above, this time where $c = c(x)$ may take either sign. Assume that there exists a positive supersolution $v(x)$, namely a function $v \in C^2(\bar{D})$ such that

$$Lv \leq 0 \ , \qquad v(x) > 0 \ .$$

Consider a subsolution $u(x) \in C^2(\bar{D})$. Show that the ratio $w(x) = u(x)/v(x)$ cannot have a nonnegative interior maximum x_0

$$0 \leq w(x_0) = M := \max_{\bar{D}}(w(x))$$

unless $u(x)$ is a multiple of $v(x)$, and both are solutions of $Lu = 0$.

Problem 3. (Green's functions for $\mathbb{R}^n_+$ and for $B_1(0) \subseteq \mathbb{R}^n$)

(a) Construct the Green's function $G(x, y)$ for the upper half space $\mathbb{R}^n_+ := \{x = (x_1, \ldots x_n) \in \mathbb{R}^n, \ x_n > 0\}$, using the fundamental solution $\Gamma(|x-y|)$ and the method of reflection. Reflections such as this are conformal transformations. The function $D(x, y) := \partial_{N_y} G(x, y)|_{x \in \mathbb{R}^n_+, \ y \in \partial \mathbb{R}^n_+}$ is called the *Poisson kernel.* Give an explicit expression for $D(x, y)$.

(b) The Green's function for the ball $B_a(0) \subseteq \mathbb{R}^n$ of radius a is also an explicit construction. Given $y \in B_a(0)$, its *inversion* in the sphere $S_a(0)$ is given by

$$y^* = \frac{a^2}{|y|^2} y \ , \quad \text{for which } |y^*| > a \ .$$

Inversions such as this are conformal transformations. Show that the sphere $S_a(0)$ can be characterized as the set of points x such that

$$\frac{|x-y^*|}{|x-y|} = \frac{a}{|y|}$$

a constant. Secondly, show that the fundamental solutions $\Gamma(x,y)$ and $\Gamma(x,y^*)$, with singularity at $x=y$ and $x=y^*$ respectively, satisfy the relation (when $n>2$)

$$\Gamma(x,y) - \frac{a^{n-2}}{|y|^{n-2}}\Gamma(x,y^*) = 0 \quad \text{when } x \in S_a(0) \ .$$

Furthermore, notice that the term $\frac{a^{n-2}}{|y|^{n-2}}\Gamma(x,y^*)$ is nonsingular for $x \in B_a(0)$, as the denominator vanishes only for $|x|>a$. Conclude that the Green's function for $B_a(0)$ is given by

$$G(x,y) = \Gamma(x,y) - \frac{a^{n-2}}{|y|^{n-2}}\Gamma(x,y^*) \ .$$

(c) Derive the Poisson kernel $D(x,y)$ for the ball $B_a(0)$ from the expression above for the Green's function.

Problem 4. (conformal maps and Green's functions on $D \subseteq \mathbb{R}^2$) The setting of this problem is in domains $D \in \mathbb{R}^2 = \mathbb{C}^1$, their Green's functions, and their expression under conformal mappings. Consider a domain $D \subseteq \mathbb{C}$ and a conformal mapping $z = x_1 + ix_2 \mapsto w(z) = x'_1 + ix'_2$ (a conformal mapping is an analytic function such that $w : D \to D'$ is one-to-one and nondegenerate in that the Jacobian $\partial_z w \neq 0$).
(a) If $u(x')$ is a harmonic function on the domain D', namely $\Delta_{x'} u = 0$, show that $u(x) = u(w(z))$, $z = x_1 + ix_2 \in D$ is a harmonic function on D, that is $\Delta_x u = 0$.
(b) For $w \in \mathbb{R}^2$ the fundamental solution in $\mathbb{R}^2$ is given by

$$\Gamma(x',y') = \frac{1}{2\pi}\log(|x'-y'|) \ .$$

Show that for $(x,y) \in D \times D$, a fundamental solution is given by

$$H(x,y) := \Gamma(w(x_1+ix_2), w(y_1+iy_2)) \ .$$

(c) Suppose that $G_{D'}(x',y')$ is the Green's function for a domain $D' \subseteq \mathbb{R}^2$, and that $w : D \to D'$ is a conformal mapping. Show that the Green's function on D is given by

$$G_D(x,y) = G_{D'}(x',y') \ ,$$

where $x' = w(x)$ and $y' = w(y)$.