Math 742. Semester 2, 2015-2016
Problem Set #5

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Problem 1. (Pohozaev identity) This problem concerns the virial identity on star-shaped domains, and a constraint on existence of solutions to certain nonlinear problems.

A domain $D \subseteq \mathbb{R}^n$ is *star-shaped* with respect to a point x_0 (which by translation may as well be taken to be $x_0 = 0$) if for each $x \in D$ the ray $\{\lambda x : 0 \le \lambda \le 1\}$ also lies in D. For such domains it is a geometric fact that for all $x \in \partial D$ then $x \cdot N \ge 0$, where N is the outward normal to the boundary.

(a): Consider solutions of the Poisson problem

$$\Delta u = h \ , \quad x \in D \ , \qquad u(x) = 0 \ , \quad x \in \partial D \ ,$$

supposing that $u \in C^2(\overline{D})$. The vector field of infinitesimal dilations is $x \cdot \partial_x = \sum_{j=1}^n x_j \partial_{x_j}$. Integrating this equation against $x \cdot \partial_x u$, prove the *virial identity*

$$\begin{aligned} \frac{n-2}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_{\partial D} |\nabla u|^2 (x \cdot N) \, dS \\ = - \int_D (nh + x \cdot \nabla h) u \, dx \ . \end{aligned}$$

Note that on star-shaped domains the quantity $x \cdot N$ is nonnegative.

(b): Now replace the term $h(x)$ by a nonlinear function of u, namely consider C^2 solutions of

$$\Delta u = -|u|^{p-1} u \ , \quad u(x) = 0 \quad x \in \partial D \ . \tag{1}$$

Show that in this situation the RHS of the virial identity can be written as

$$RHS = \frac{n}{p+1} \int_D |u|^{p+1} \, dx \ .$$

(c): Together the two identities above state that

$$\frac{n-2}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_{\partial D} |\nabla u|^2 (x \cdot N) \, dS = \frac{n}{p+1} \int_D |u|^{p+1} \, dx \ .$$

Show by other means that a C^2 solution of (1) satisfies

$$\int_D |\nabla u|^2 \, dx = \int_D |u|^{p+1} \, dx \ .$$

Since both identites must hold, then

$$\frac{n-2}{2}\int_D |u|^{p+1}\,dx \le \frac{n}{p+1}\int_D |u|^{p+1}\,dx\ ,$$

with the implication that when $p > \frac{n+2}{n-2}$ then the only solution is $u = 0$.

Problem 2. (path integration) We want to consider solutions of an evolution equation for $v \in H$, where H is a Hilbert space;

$$\partial_t v = \mathbf{A} v\ , \quad v(0) = v_0 \in H\ , \tag{2}$$

and a representation of solutions of a 'perturbation' of this equation given by

$$\partial_t v = (\mathbf{A} + \mathbf{B}) v\ , \quad v(0) = v_0 \in H\ . \tag{3}$$

Consider the situation where $\mathbf{A}$ is self-adjoint operator on the Hilbert space H such that the operator semigroup $\{e^{t\mathbf{A}} : t \in \mathbb{R}^+\}$ is bounded uniformly, namely $\|\mathbf{A}v\|_H \le C_A \|v\|_H$ (the operator $\mathbf{A}$ itself may be unbounded). Suppose that $\mathbf{B}$ is a bounded operator on H.

This exercise is to show that one can construct the operator $e^{(\mathbf{A}+\mathbf{B})t}$ as a perturbation expansion, the *Dyson series*, which expresses the principles of Feynman path expansions.

(a): Use Duhamel's principle to rewrite the solution of equation (3) as an integral equation

$$v(t) = e^{t\mathbf{A}} v_0 + \int_0^t e^{(t-s)\mathbf{A}} (\mathbf{B} v(s))\,ds\ . \tag{4}$$

Denote

$$\mathbf{L}_B v := \int_0^t e^{(t-s)\mathbf{A}} (\mathbf{B} v)\,ds\ ,$$

and use the equation (4) to write $v(t)$ as a formal series

$$v(t) = \sum_{m \in \mathbb{N}} \mathbf{L}_B^m (e^{t\mathbf{A}} v_0)\ . \tag{5}$$

(c): Prove that

$$\| \int_0^t e^{(t-s)\mathbf{A}} Bv\,ds \|_H \le C_A \int_0^t \|\mathbf{B}v\|_H \le C_A C_B \int_0^t \|v\|_H\,ds\ .$$

Now suppose for purposes of induction that

$$\|v(t)\|_H \le \frac{(C_A C_B t)^r}{r!}$$

prove that

$$\| \int_0^t e^{(t-s)\mathbf{A}} Bv(s)\,ds \|_H \le \frac{(C_A C_B t)^{r+1}}{(r+1)!}\ ,$$

hence demonstrating the convergence of the series that you constructed formally in (5).

Problem 3. (initial - boundary value problems for the heat equation) The heat kernel in one space dimension is of course

$$H(t, x-y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \ .$$

(a): For periodic data $u_0(x + 2\pi k) = u_0(x)$ for all $k \in \mathbb{N}$, show that

$$\int_{-\infty}^{+\infty} H(t, x-y) u_0(y)\, dy = \int_0^{2\pi} H_p(t, x-y) u_0(y)\, dy \ ,$$

where

$$H_p(t, x-y) = \sum_{k=-\infty}^{+\infty} H(t, x-y+2\pi k) \ ;$$

show that this sum converges and that it satisfies the appropriate partial differential equations and boundary conditions.

(b): For the heat equation on the interval $[0, 2\pi]$ with Dirichlet boundary conditions, the heat kernel can be described as follows:

$$H_D(t, x, y) = H_p(t, x-y) - H_p(t, x+y) \ .$$

Show that this expression satisfies the appropriate partial differential equations and boundary conditions.

(c): An integral operator K on functions of $x \in [0, 2\pi]$ with kernel function $k(x, y)$ has a trace given by

$$\mathrm{tr}\ (K) = \int_0^{2\pi} k(x, x)\, dx \ .$$

Define the Laplacian operator with periodic boundary conditions to be Δ_p, and that with Dirichlet boundary conditions to be Δ_D. Calculate tr $(e^{-t\Delta_p})$ and show that it is finite for each $t > 0$ but diverges as $t \to 0+$.

Calculate tr $(e^{-t\Delta_D})$ and show that it is also finite for each $t > 0$ but that it also diverges as $t \to 0+$.

Finally, compute the trace of the difference,

$$M(t) := \mathrm{tr}\ (e^{-t\Delta_p} - e^{-t\Delta_D})$$

and show that the limit $\lim_{t\to 0+} M(t)$ is finite. What is the limit?

Problem 4. (maximum principle for vorticity) The incompressible Navier – Stokes equations in two dimensions for the velocity field $u = (u_1, u_2)(t, x)$ are

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u$$
$$\nabla \cdot u = 0 \ .$$

We will consider the case of $u(t, x + 2\pi k) = u(t, x)$ for all integer pairs $(k_1, k_2) \in \mathbb{N}^2$, which is to say a doubly periodic domain D with no boundary.

(a): Derive an expression for the velocity field $u(\cdot, x)$ in terms of the vorticity $\omega(\cdot, x)$ (the Biot – Savart law), and show that if $\omega \in C(D)$ then $u \in C^\alpha(D)$ for all $0 < \alpha < 1$.

(b): Derive an evolution equation for the vorticity $\omega(t, x) = \partial_{x_1} u_2 - \partial_{x_2} u_1$ in terms of the incompressible velocity field $u(t, x)$, for given initial vorticity $\omega_0(x) \in C(D)$.

(c): Show that the vorticity satisfies the maximum principle; namely show that for all $(t, x) \in \mathbb{R}^+ \times D$

$$\min_{x \in D}(\omega_0(x)) \leq \omega(t, x) \leq \max_{x \in D}(\omega_0(x))$$

and if ever either of the extrema is achieved at some x_0 for some time $t > 0$ then the solution is constant.