Math 742. Semester 2, 2015-2016
Problem Set #5

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Problem 1. (Pohozaev identity) This problem concerns the virial identity on star-shaped domains, and a constraint on existence of solutions to certain nonlinear problems.

A domain $D \subseteq \mathbb{R}^n$ is star-shaped with respect to a point $x_0$ (which by translation may as well be taken to be $x_0 = 0$) if for each $x \in D$ the ray $\{\lambda x : 0 \leq \lambda \leq 1\}$ also lies in $D$. For such domains it is a geometric fact that for all $x \in \partial D$ then $x \cdot N \geq 0$, where $N$ is the outward normal to the boundary.

(a): Consider solutions of the Poisson problem
\[ \Delta u = h , \quad x \in D , \quad u(x) = 0 , \quad x \in \partial D , \]
supposing that $u \in C^2(D)$. The vector field of infinitesimal dilations is $x \cdot \partial_x = \sum_{j=1}^n x_j \partial_{x_j}$.

Integrating this equation against $x \cdot \partial_x u$, prove the virial identity
\[
\frac{n - 2}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_{\partial D} |\nabla u|^2 (x \cdot N) \, dS = -\int_D (nh + x \cdot \nabla h) u \, dx .
\]

Note that on star-shaped domains the quantity $x \cdot N$ is nonnegative.

(b): Now replace the term $h(x)$ by a nonlinear function of $u$, namely consider $C^2$ solutions of
\[ \Delta u = -|u|^{p-1} u , \quad u(x) = 0 \quad x \in \partial D . \quad (1) \]

Show that in this situation the RHS of the virial identity can be written as
\[
RHS = \frac{n}{p + 1} \int_D |u|^{p+1} \, dx .
\]

(c): Together the two identities above state that
\[
\frac{n - 2}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_{\partial D} |\nabla u|^2 (x \cdot N) \, dS = \frac{n}{p + 1} \int_D |u|^{p+1} \, dx .
\]

Show by other means that a $C^2$ solution of $(1)$ satisfies
\[
\int_D |\nabla u|^2 \, dx = \int_D |u|^{p+1} \, dx .
\]
Since both identities must hold, then
\[ \frac{n - 2}{2} \int_D |u|^{p+1} \, dx \leq \frac{n}{p+1} \int_D |u|^{p+1} \, dx , \]
with the implication that when \( p > \frac{n+2}{n-2} \) then the only solution is \( u = 0 \).

**Problem 2.** (path integration) We want to consider solutions of an evolution equation for \( v \in H \), where \( H \) is a Hilbert space;
\[ \partial_t v = Av , \quad v(0) = v_0 \in H , \quad (2) \]
and a representation of solutions of a ‘perturbation’ of this equation given by
\[ \partial_t v = (A + B)v , \quad v(0) = v_0 \in H . \quad (3) \]
Consider the situation where \( A \) is self-adjoint operator on the Hilbert space \( H \) such that the operator semigroup \( \{ e^{tA} : t \in \mathbb{R}^+ \} \) is bounded uniformly, namely \( \|Av\|_H \leq C_A\|v\|_H \) (the operator \( A \) itself may be unbounded). Suppose that \( B \) is a bounded operator on \( H \).

This exercise is to show that one can construct the operator \( e^{(A+B)t} \) as a perturbation expansion, the *Dyson series*, which expresses the principles of Feynman path expansions.

**(a):** Use Duhamel’s principle to rewrite the solution of equation (3) as an integral equation
\[ v(t) = e^{tA}v_0 + \int_0^t e^{(t-s)A}(Bv(s)) \, ds . \quad (4) \]
Denote
\[ L_Bv := \int_0^t e^{(t-s)A}(Bv) \, ds , \]
and use the equation (4) to write \( v(t) \) as a formal series
\[ v(t) = \sum_{m \in \mathbb{N}} L^m_B(e^{tA}v_0) . \quad (5) \]

**(c):** Prove that
\[ \| \int_0^t e^{(t-s)A}Bv \, ds \|_H \leq C_A \int_0^t \|Bv\|_H \leq C_A C_B \int_0^t \|v\|_H \, ds . \]
Now suppose for purposes of induction that
\[ \|v(t)\|_H \leq \frac{(C_A C_B t)^r}{r!} \]
prove that
\[ \| \int_0^t e^{(t-s)A}Bv(s) \, ds \|_H \leq \frac{(C_A C_B t)^{r+1}}{(r+1)!} , \]
hence demonstrating the convergence of the series that you constructed formally in (5).

**Problem 3.** (initial - boundary value problems for the heat equation) The heat kernel in one space dimension is of course

\[ H(t, x - y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}. \]

(a): For periodic data \( u_0(x + 2\pi k) = u_0(x) \) for all \( k \in \mathbb{N} \), show that

\[ \int_{-\infty}^{+\infty} H(t, x - y)u_0(y) \, dy = \int_0^{2\pi} H_p(t, x - y)u_0(y) \, dy, \]

where

\[ H_p(t, x - y) = \sum_{k=-\infty}^{+\infty} H(t, x - y + 2\pi k); \]

show that this sum converges and that it satisfies the appropriate partial differential equations and boundary conditions.

(b): For the heat equation on the interval \([0, 2\pi]\) with Dirichlet boundary conditions, the heat kernel can be described as follows:

\[ H_D(t, x, y) = H_p(t, x - y) - H_p(t, x + y). \]

Show that this expression satisfies the appropriate partial differential equations and boundary conditions.

(c): An integral operator \( K \) on functions of \( x \in [0, 2\pi] \) with kernel function \( k(x, y) \) has a trace given by

\[ \text{tr} \,( K) = \int_0^{2\pi} k(x, x) \, dx. \]

Define the Laplacian operator with periodic boundary conditions to be \( \Delta_p \), and that with Dirichlet boundary conditions to be \( \Delta_D \). Calculate \( \text{tr} \,( e^{-t\Delta_p}) \) and show that it is finite for each \( t > 0 \) but diverges as \( t \to 0^+ \).

Calculate \( \text{tr} \,( e^{-t\Delta_D}) \) and show that it is also finite for each \( t > 0 \) but that it also diverges as \( t \to 0^+ \).

Finally, compute the trace of the difference,

\[ M(t) := \text{tr} \,( e^{-t\Delta_p} - e^{-t\Delta_D}) \]

and show that the limit \( \lim_{t \to 0^+} M(t) \) is finite. What is the limit?

**Problem 4.** (maximum principle for vorticity) The incompressible Navier – Stokes equations in two dimensions for the velocity field \( u = (u_1, u_2)(t, x) \) are

\[ \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u \]
\[ \nabla \cdot u = 0. \]
We will consider the case of $u(t, x + 2\pi k) = u(t, x)$ for all integer pairs $(k_1, k_2) \in \mathbb{N}^2$, which is to say a doubly periodic domain $D$ with no boundary.

(a): Derive an expression for the velocity field $u(\cdot, x)$ in terms of the vorticity $\omega(\cdot, x)$ (the Biot–Savart law), and show that if $\omega \in C(D)$ then $u \in C^\alpha(D)$ for all $0 < \alpha < 1$.

(b): Derive an evolution equation for the vorticity $\omega(t, x) = \partial_{x_1} u_{2} - \partial_{x_2} u_{1}$ in terms of the incompressible velocity field $u(t, x)$, for given initial vorticity $\omega_0(x) \in C(D)$.

(c): Show that the vorticity satisfies the maximum principle; namely show that for all $(t, x) \in \mathbb{R}^+ \times D$

$$\min_{x \in D}(\omega_0(x)) \leq \omega(t, x) \leq \max_{x \in D}(\omega_0(x))$$

and if ever either of the extrema is achieved at some $x_0$ for some time $t > 0$ then the solution is constant.