Solitary water wave interactions

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Rogue waves
ICMS – Edinburgh
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Collaborators

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Diane Henderson   Pennsylvania State University

Joe Hammack -2004

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portrait of J. C. Maxwell (1831 - 1879)

03/10/05 http://www.egr.msu.edu/~bohnsac3/nano/figures/Maxwell.jpg #1

Abstract

- Discussion of the classical problem of interactions of solitary water waves
  - symmetric collisions (interaction with a seawall)
  - antisymmetric counterpropagating interactions
  - overtaking wave interactions
- Main results
  - accurate numerical simulations
  - Is the theory of Su & Mirie (1980) complete?
  - relate ΔE (energy loss) and ΔS (amplitude change).

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Outline

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Equations of potential flow

- Euler's equations

$$\Delta\varphi = 0\,, \quad N\cdot\nabla\varphi = 0 \quad \text{on} \quad y = -h$$

Nonlinear boundary conditions on the free surface

$$\left.\begin{aligned} \varphi_t + \tfrac{1}{2}(\nabla\varphi)^2 + g\eta = 0 \\ \eta_t + \eta_x\,\varphi_x - \varphi_y = 0 \end{aligned}\right\} \quad \text{on} \quad y = \eta(x,t)\,,$$

- Conservation laws

$$M = \int \eta(x)\,dx\,, \quad \text{mass.} \quad I = \int\int_{-h}^{\eta} \nabla_x\varphi\,dydx\,, \quad \text{momentum.}$$
$$H = \int\int_{-h}^{\eta} \tfrac{1}{2}|\nabla\varphi|^2\,dydx + \tfrac{g}{2}\int \eta^2(x)\,dx \quad \text{energy.}$$

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Solitary water waves

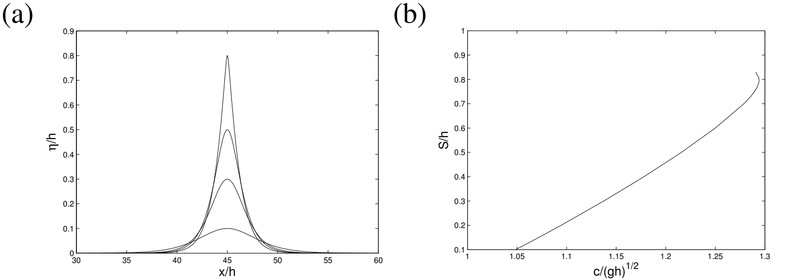
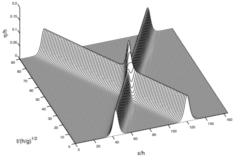


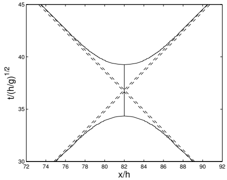
Figure: (a) Solitary waves of height $S/h = 0.1, 0.3, 0.5, 0.8$. (b) Bifurcation diagram. Computed by the modified Tanaka's method.

- Head-on collision, case $S/h = 0.1$

(a)

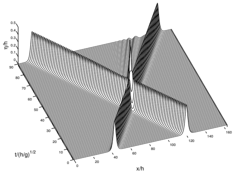


(b)

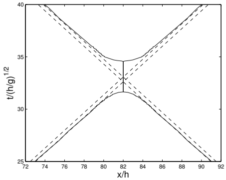


- case $S/h = 0.4$

(a)

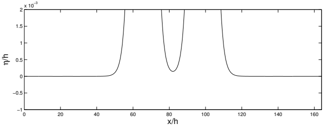


(b)

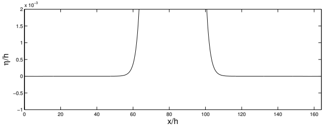


Residual after collision , case $S/h = 0.1$

(a)



(b)



(c)

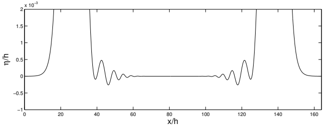


Figure: $S/h = 0.1$ at (a) $t/\sqrt{h/g} = 21$ (before collision), (b) 45 (during collision), (c) 90 (after collision). Vertical scale is magnified.

Symmetric interactions: separation after long time

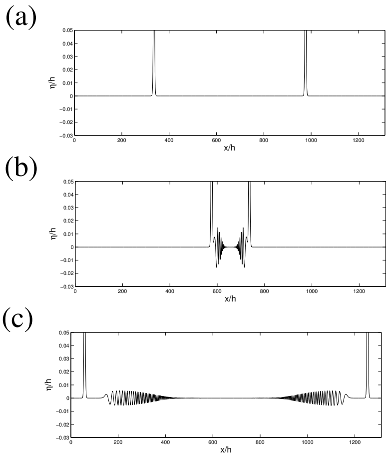


Figure: Head-on collision $S/h = 0.4$, at (a) $t/\sqrt{h/g} = 0$, (b) 340, (c) 780.

Run-up in symmetric interactions: $S/h = 0.4$

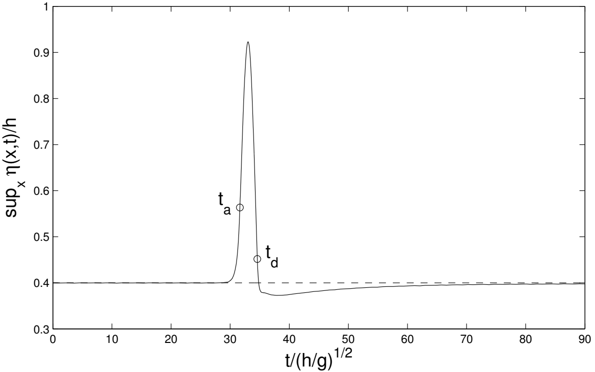


Figure: Time evolution of the amplitude $\|\eta(x,t)\|_{L^\infty(\mathbb{R}_x)}$, head-on collision, height $S/h = 0.4$. Attachment and detachment times t_a and t_d circles.

Run-up and wall residence time of symmetric interactions

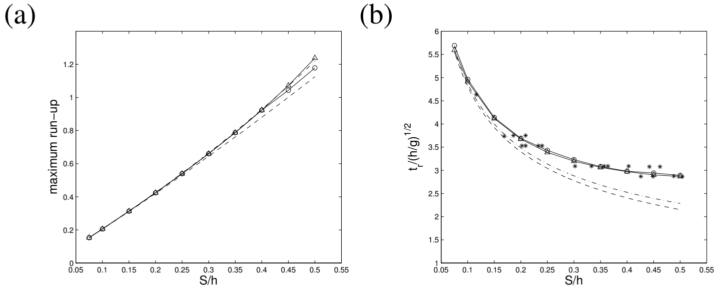


Figure: Maximum run-up as a function of incident wave height:
Our numerical results (solid line-circles),
Cooker, Weidman and Bale (1997) numerical results (solid line-triangles),
Su and Mirie (1980) perturbation results: second-order (dashed line) and third-order (dotted-dashed line).

Comparison with PSU wavechannel experiments

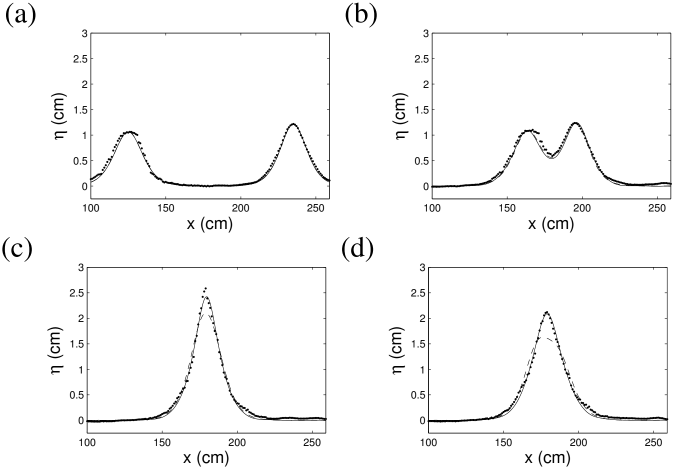


Figure: $S_1 = 1.217, S_2 = 1.063$ (cm) at (a) $t = 18.29993$, (b) 18.80067, (c) 19.05257, (d) 19.10173; numerical results (solid line), experimental

Comparison with PSU wavechannel experiments (continued)

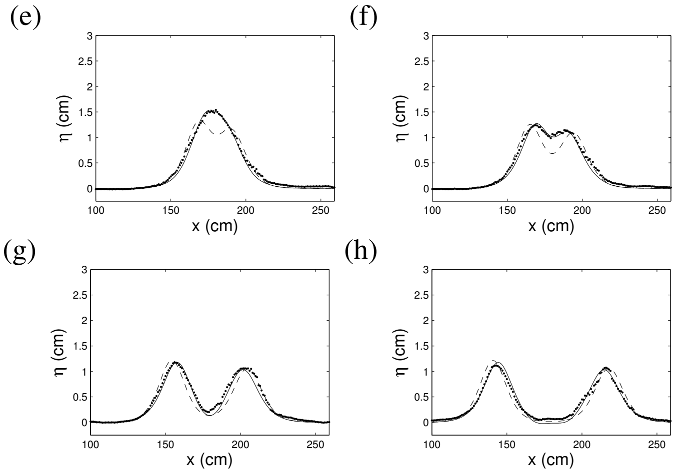


Figure: (e) 19.15088, (f) 19.19389, (g) 19.32905, (h) 19.50109; numerical results (solid line), experimental results (dots), sum of two KdV solitons

- Zakharov's Hamiltonian (1968) for $\eta(x,t)$, $\xi(x,t) = \varphi(x, \eta(x,t), t)$

$$H(\eta, \xi) = \frac{1}{2} \int_{-\infty}^{\infty} \xi G(\eta)\xi + g\eta^2 \, dx$$

- Dirichlet – Neumann operator

$$G(\eta)\xi = \sqrt{1+\eta_x^2}\,(N \cdot \nabla\varphi)\Big|_{y=\eta}$$

- Hamilton's canonical equations

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix} .$$

NB: The variation of the kinetic energy H_{ke} with respect to the domain $\eta(x)$, J. Hadamard (1911)

Surface spectral methods

- Equations of motion

$$
\begin{aligned}
\eta_t &= G(\eta)\xi \\
\xi_t &= \frac{-1}{2(1+\eta_x^2)} \left[\xi_x^2 - (G(\eta)\xi)^2 - 2\eta_x \, \xi_x \, G(\eta)\xi\right] - g\eta
\end{aligned}
$$

- spectral representation, notation $D = -i\partial_x$

$$
\begin{aligned}
G_0 &= D \tanh(hD)\,, \quad G_1 = D\eta D - G_0 \eta G_0 \\
G_2 &= \frac{1}{2}(G_0 D\eta^2 D - D^2\eta^2 G_0 - 2G_0\eta G_1
\end{aligned}
$$

Overtaking interactions: experimental and numerical data

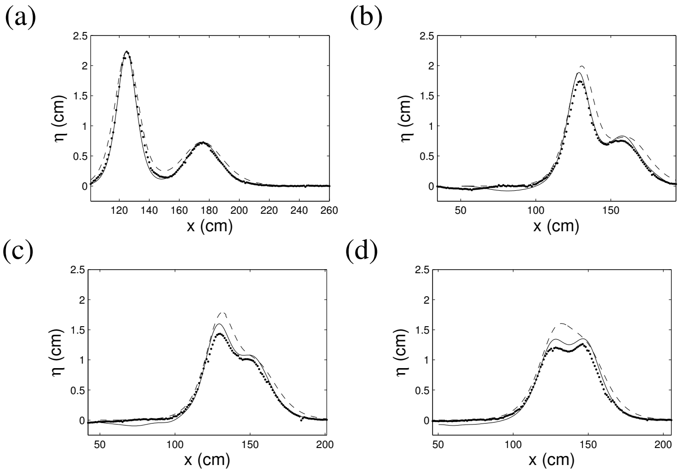


Figure: $S_1 = 2.295$, $S_2 = 0.730$ (cm) at (a) $t = 2.90304$, (b) 5.50196, (c) 6.40513, (d) 7.05025; Reference frame with zero relative speed for the

Experimental and numerical data (continued)

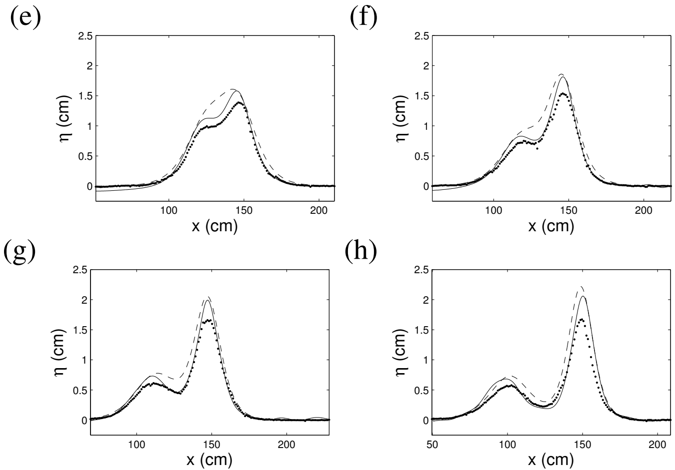


Figure: (e) 7.60014, (f) 8.50024, (g) 9.50478, (h) 11.30191: numerical results (solid line), experimental results (dots), KdV two-soliton solution

- Overtaking collision: $S_1/h = 0.4$, $S_2/h = 0.3$

(a) (b)

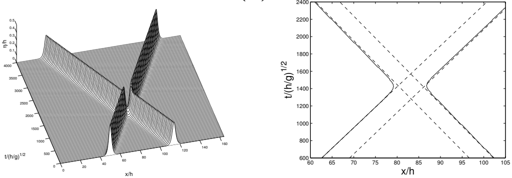
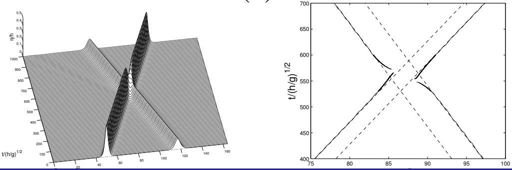


Figure: Amplitudes $S_1/h = 0.4$, $S_2/h = 0.3$: Amplitudes after collision: $S_1^+/h = 0.4004$, $S_2^+/h = 0.2999$ at $t/\sqrt{h/g} = 4000$.

- Case $S_1/h = 0.4$, $S_2/h = 0.133$

(a) (b)



- Overtaking collision: $S_1/h = 0.4$, $S_2/h = 0.113$

(a) (b)

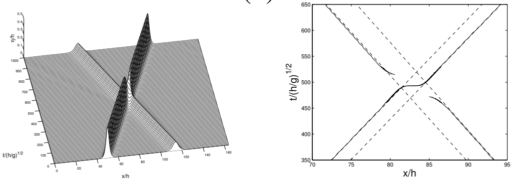
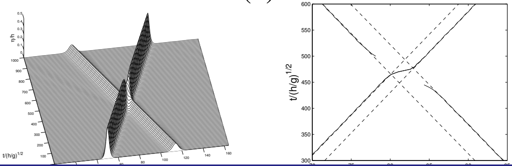


Figure: Amplitudes after collision: $S_1^+/h = 0.4001$, $S_2^+/h = 0.1129$ at $t/\sqrt{h/g} = 1000$.

- Case $S_1/h = 0.4$, $S_2/h = 0.1$

(a) (b)



The maximum elevation

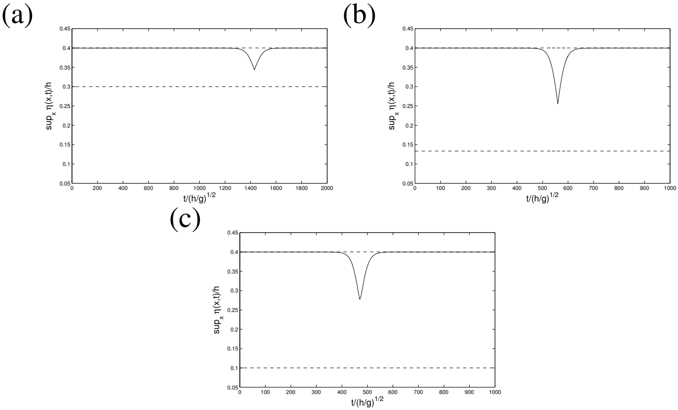


Figure: Time evolution of the amplitude $\|\eta(x,t)\|_{L^\infty(\mathbb{R}_x)}$ for the overtaking collision of two solitary waves of height (a) $S_1/h = 0.4$, $S_2/h = 0.3$, (b) $0.4, 0.1333$ and (c) $0.4, 0.1$.

The residual

In case $S_1/h = 0.4$, $S_2/h = 0.3$

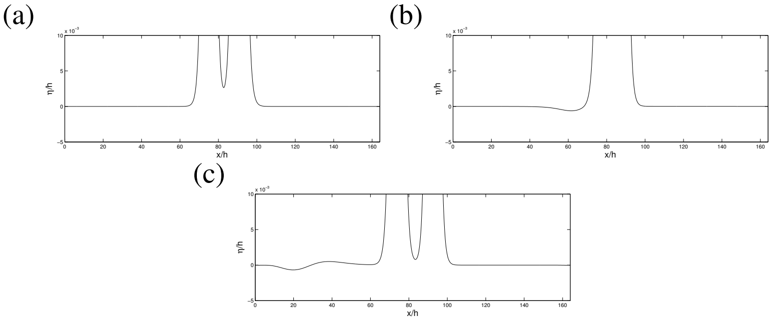


Figure: Overtaking collision of two solitary waves of height $S_1/h = 0.4$, $S_2/h = 0.3$ at (a) $t/\sqrt{h/g} = 1190$ (before collision), (b) 1490 (during collision), (c) 1740 (after collision). The vertical scale is magnified in order to observe the dispersive trailing wave generated after the collision.

The conservation laws

- Before the interaction

$$
\begin{aligned}
M_T &= m(S_1) + m(S_2)\ , \qquad I_T = \mu(S_1) + \mu(S_2) \\
E_T &= e(S_1) + e(S_2)\ .
\end{aligned}
$$

- After the interaction

$$
\begin{aligned}
M_T &= m(S_1^+) + m(S_2^+) + m_R \qquad I_T = \mu(S_1^+) + \mu(S_2^+) + \mu_R \\
E_T &= e(S_1^+) + e(S_2^+) + e_R\ .
\end{aligned}
$$

- Taking the difference and using the conservation laws

$$
\begin{aligned}
m_1'\Delta S_1 + m_2'\Delta S_2 &= m_R \qquad \mu_1'\Delta S_1 + \mu_2'\Delta S_2 = \mu_R \\
e_1'\Delta S_1 + e_2'\Delta S_2 &= e_R\ .
\end{aligned}
$$

In the symmetric case

- In particular the symmetric case, $S := S_1 = S_2$, implying by symmetry $\Delta S := \Delta S_1 = \Delta S_2$, $I_T = 0$ and $\mu_R = 0$.

$$2m'\Delta S = m_R \;, \qquad 2e'\Delta S = e_R$$

- For small symmetric interactions $e(S) \sim S^{3/2}$

$$3S^{1/2}\Delta S = e_R$$

- The theory of Su & Mirie states

$$\eta_R \sim S^3 \;, \qquad \Delta S = o(S^3)$$

Symmetric interactions - data table

S/h	S^+/h	$(S-S^+)/h$	E_T	e_R $(\times 10^3)$	$\frac{e_R}{E_T}$ $(\times 10^3)$
0.025	0.02490	0.00010	0.011	0.092	8.358
0.05	0.04983	0.00017	0.034	0.192	5.564
0.075	0.07476	0.00024	0.065	0.338	5.174
0.1	0.09964	0.00036	0.102	0.598	5.865
0.15	0.14930	0.00070	0.191	1.378	7.203
0.2	0.19892	0.00108	0.299	2.517	8.403
0.25	0.24859	0.00141	0.425	3.809	8.968
0.3	0.29834	0.00166	0.565	5.400	9.562
0.35	0.34788	0.00212	0.718	7.791	10.855
0.4	0.39738	0.00262	0.882	8.817	9.999
0.45	0.44534	0.00466	1.054	16.323	15.488
0.5	0.49311	0.00689	1.231	24.712	20.067

E_T vs. S/h

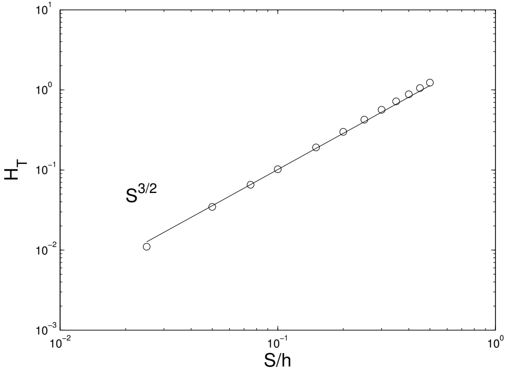


Figure: Total energy E_T vs. wave amplitude S/h: numerical results (circles), power law $(S/h)^{3/2}$ (solid line).

ΔS vs. S/h

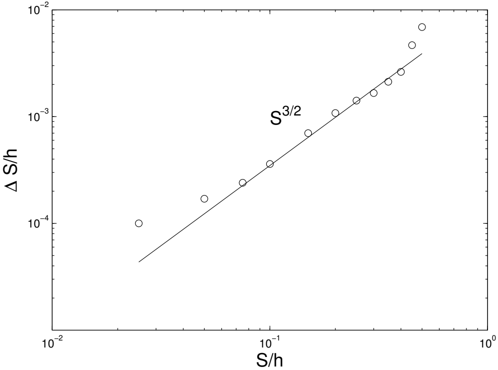


Figure: Change in amplitude $\Delta S/h = (S - S^+)/h$ vs. wave amplitude S/h: numerical results (circles), power law $(S/h)^{3/2}$ (solid line).

Residual energy vs. S/h

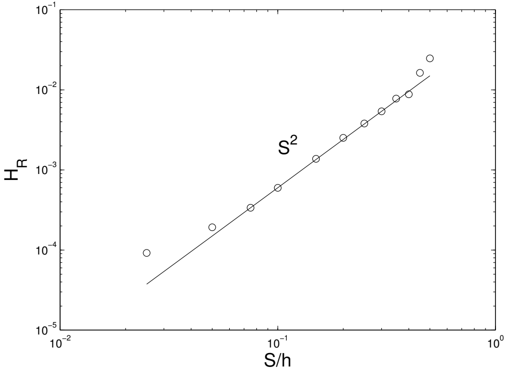


Figure: Energy of the residual e_R vs. nondimensional wave amplitude S/h: numerical results (circles), power law $(S/h)^2$ (solid line).

Thank you

Theorem: (Schneider & Wayne (2000), Bona, Colin & Lannes (2005), Wright (2005)

- Two decoupled KdV equations at $\mathcal{O}(S)$

$$-2\partial_T q^- = \frac{1}{3}\partial_{X_-}^3 q^- + 3q^-\partial_{X_-} q^- \,, \quad 2\partial_T q^+ = \frac{1}{3}\partial_{X_+}^3 q^+ + 3q^+\partial_{X_+} q^+$$

- Three equations at $\mathcal{O}(S^2)$

$$\begin{aligned} -2\partial_T f^- &= \frac{1}{3}\partial_{X_-}^3 f^- + 3\partial_{X_-}(q^- f^-) + J^- \\ 2\partial_T f^+ &= \frac{1}{3}\partial_{X_+}^3 f^+ + 3\partial_{X_+}(q^+ f^+) + J^+ \\ \partial_\tau^2 p - \partial_X^2 p &= 3\partial_X^2\Big(q^-(X-\tau,\varepsilon^2\tau)q^+(X+\tau,\varepsilon^2\tau)\Big) \,. \end{aligned}$$

- Estimate of the error are $\|\mathcal{E}(x,t)\|_{H^s} \leq C(S/h)^{11/4}$.