Lagrangian and resonant invariant tori for Hamiltonian systems with infinitely many degrees of freedom

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Hamiltonian systems

- Hamiltonian vector field on a *phase space*. $v \in \mathcal{H}$ a Hilbert space

$$\partial_t v = X_H(v) = \quad J\mathrm{grad}_v H(v) \ , \qquad v(x,0) = v^0(x) \ , \quad (1)$$

- Symplectic form

$$\omega(X,Y) = \langle X, J^{-1}Y\rangle_{\mathcal{H}} \ , \quad J^T = -J \ .$$

- The flow $\qquad v(x,t) = \varphi_t(v^0(x))$
- Interest in orbits where

$$\overline{\{\varphi_t(v^0) \ : \ t \in \mathbb{R}\}} = \mathbb{T}^m$$

  an m-dimensional torus. This gives stable motions of (1).
- Invariant tori of maximal dimension are Lagrangian tori, $m = \infty$

lattice nonlinear Schrödinger equation

- Hamiltonian system posed on a lattice $k \in \mathbb{Z}^+$

$$\frac{1}{i}\partial_t q_k = \mu_k q_k + |q_k|^2 q_k + \varepsilon(\Delta q)_k \tag{2}$$

  with $q_0 = 0$, Dirichlet boundary conditions.
  Phase space is $\mathcal{H} = \ell^2_{\mathbb{C}}(\mathbb{Z}^+)$, and the symplectic form is
  $\omega = i\sum_k dq_k \wedge d\overline{q}_k$
- The Hamiltonian is $H(q) : \mathcal{H} \mapsto \mathbb{R}$

$$\begin{aligned} H &= \sum_k \mu_k |q_k|^2 + \tfrac{1}{2}|q_k|^4 + \varepsilon \sum_k (\overline{q}_k q_{k+1} + q_k \overline{q}_{k+1}) \\ &= N + \varepsilon P \end{aligned} \tag{3}$$

Outline

Lattice nonlinear Schrödinger equations

Lagrangian invariant tori for lattice Schrödinger equations

A variational formulation for invariant tori

Details of the KAM iteration

Resonant situations

Normal form

- Integrable unperturbed problem, when $\varepsilon = 0$
  Uncoupled anharmonic oscillators

$$\frac{1}{i}\partial_t q_k = \mu_k q_k + |q_k|^2 q_k \ , \qquad k \in \mathbb{Z}^+ \tag{4}$$

- Solutions of the unperturbed flow $\varphi_t^0(q)$

$$q_k(t) = \sqrt{I_k} e^{i(\mu_k + I_k)t} \ , \qquad \Omega_k^0(I) = \mu_k + I_k$$

- gauge invariance and the ℓ^2-norm $K := \|q\|_{\ell^2}^2$: for $\theta \in \mathbb{T}^1$

$$e^{i\theta}\varphi_t(q) = \varphi_t(e^{i\theta}q)$$

From the fact that $\{H, K\} = 0$ are Poisson commuting

Lagrangian tori

Theorem (WC & J. Geng (2008))

Let $\mu_k = k$. There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there exists a Cantor-like set $\mathcal{O}_\varepsilon \subseteq \ell^\infty(\mathbb{Z}^+)$ such that for $I \in \mathcal{O}_\varepsilon$ there is an invariant Lagrangian torus $\mathbb{T} \subseteq \ell^2_{\mathbb{C}}(\mathbb{Z}^+)$ for (4).
The torus $\mathbb{T}$ is of the form

$$\begin{aligned} q_k(t) &= \sqrt{I_k} e^{i\omega_k(I)t} + \mathcal{O}(\sqrt{\varepsilon I_k}) \ , \\ \omega_k(I) &= \mu_k + I_k + \mathcal{O}(\varepsilon) \end{aligned}$$

The measure of the set $\mathcal{O}_\varepsilon$ is positive (in some sense) and tends to full measure as $\varepsilon \to 0$.

The point is that there are no external parameters.

parameters

- We use the classical KAM iteration scheme of iteration of symplectic transformations.
- Parameters are used in order to avoid near-resonances. In this case the action variables $(I_k)_{k\in\mathbb{Z}^+} \in \ell^\infty(\mathbb{Z}^+)$ play the rôle of parameters.
- In order to do this, the KAM iteration scheme has an augmented number of nonresonance conditions. Namely, at the ν^{th} step $\vec{\omega}^{(\nu)} = (\omega_1, \dots, \omega_\nu)$ the first ν-many tangential frequencies satisfy

$$|\langle k, \vec{\omega}^{(\nu)}\rangle + \langle \ell, \vec{\Omega}^{(\nu)}\rangle| \geq \frac{\gamma_\nu}{|k|^{\tau_\nu}}$$

for $k \in \mathbb{Z}^\nu$ and for $|\ell| \leq 4$.

Theorem 2

Theorem (J. Geng (2008))

A similar result for the nonlinear Schrödinger equation on $[0, 2\pi]$ *with Dirichlet or periodic boundary conditions.*

$$\frac{1}{i}\partial_t q = -\Delta q + \varepsilon h'(|q|^2) q$$

with $h(|q|^2) \simeq \pm |q|^4 + \cdots$

Other results on Lagrangian tori

- J. Fröhlich, T. Spencer & E. Wayne (1986) discrete Schrödinger equation with random potential $V(x,\omega), \omega \in \Omega$
- J. Pöschel (1990), *Small divisors with spatial structure*
- L. Chierchia & P. Perfetti (1995) Frequencies μ_k which grow rapidly.
- J. Bourgain (1996) wave equation with a potential $V(x)$;

$$\partial_t^2 u - \partial_x^2 u + V(x)u + F(u) = 0$$

- J. Pöschel (2002), smoothed NLS, with a potential $V(x)$
- J. Bourgain (2005), NLS, with a Fourier multiplier giving parameters

Extensions of the lattice NLS problem

1. linear frequencies $\mu_k = k^2$ the discrete harmonic operator, or the Fourier transform of the nonlinear Schrödinger equation on $\mathbb{S}^1$. Additionally $\mu_k = k^n$.
2. The full line problem $k \in \mathbb{Z}$, with $\mu_k \neq \mu_{-k}$ to avoid resonance. Easy extensions, taking $q_{-k} = -q_k$ odd, or $q_{-k} = q_k$. The general case is harder, but possible too.
3. Different nonlinearities and perturbations

$$H = \sum_k \mu_k |q_k|^2 + h(|q_k|^2) + \varepsilon \sum_{k,\ell} \overline{q}_k A_{k,\ell} q_\ell$$

with $A_{k\ell} = A^*_{\ell k}$, as long as

$$h(q) \simeq |q|^4 + \cdots \ , \qquad |A_{k,\ell}| \leq e^{-\rho|k-\ell|}$$

- Gauge invariance $\{q_k\}_{k\in\mathbb{Z}^+} \to \{e^{i\psi}q_k\}_{k\in\mathbb{Z}^+}$
- We may break gauge invariance, $h = h(q,\overline{q})$ with extra nonresonance conditions.

Resonant invariant tori

- Mapping of a torus $S(\theta) : \mathbb{T}^m \mapsto \mathcal{H}$
- Flow invariance $S(\theta + t\Omega) = \varphi_t(S(\theta))$
  Frequency vector $\Omega \in \mathbb{R}^m$.
- This implies that both

$$\partial_t S = J \operatorname{grad}_v H(S) \ , \quad \text{and} \quad \partial_t S = \Omega \cdot \partial_\theta S \tag{5}$$

- Problem: Solve (5) for $(S(\theta), \Omega)$.
  This is generally a small divisor problem.

  Rewrite (5) as

$$J^{-1}\Omega \cdot \partial_\theta S - \operatorname{grad}_v H(S) = 0 \ . \tag{6}$$

A variational formulation

Consider the space of mappings $S \in X := \{S(\theta) : \mathbb{T}^m \mapsto \mathcal{H}\}$.
Suppose that $m < +\infty$

- Define action functionals

$$
\begin{aligned}
I_j(S) &= \frac{1}{2}\int_{\mathbb{T}^m} \langle S, J^{-1}\partial_{\theta_j} S\rangle \, d\theta \\
\delta_S I_j &= J^{-1}\partial_{\theta_j} S
\end{aligned}
$$

  This is the moment map for *mappings*

- The average Hamiltonian

$$
\begin{aligned}
\overline{H}(S) &= \int_{\mathbb{T}^m} H(S(\theta)) \, d\theta \\
\delta_S \overline{H} &= \mathrm{grad}_v H(S)
\end{aligned}
$$

interpretation

Consider the subvariety of X defined by fixed actions

$$M_a = \{S \in X : I_1(S) = a_1, \dots I_m(S) = a_m\} \subseteq X$$

Variational principle: critical points of $\overline{H}(S)$ on M_a correspond to solutions of equation (6), with Lagrange multiplier Ω.

NB: All of $\overline{H}(S)$, $I_j(S)$ and M_a are invariant under the action of the torus $\mathbb{T}^m$; that is $\tau_\alpha : S(\theta) \mapsto S(\theta + \alpha)$, $\alpha \in \mathbb{T}^m$.

This poses several questions

- Two questions.
  1. Do critical points exist on M_a?
     Note that the following operators are degenerate on the space of mappings X:
     $$\Omega \cdot J^{-1}\partial_\theta S \, , \qquad \Omega \cdot J^{-1}\partial_\theta S - \delta_S^2 \overline{H}(0)$$
  2. How to understand questions of multiplicity of solutions?
- Proposal to address this question
  1. Use KAM or Nash – Moser methods with parameters
     Direct Nash – Moser methods rely on solutions of the linearized equations via resolvant expansions (Fröhlich – Spencer estimates)
  2. Equivariant Morse – Bott theory of critical $\mathbb{T}^m$ orbits.

prior results

Theorem (C-Q Cheng (1993))

The existence of a minimal $m = (n-1)$-dimensional resonant torus.

Hamiltonian at the ν-th KAM step

- The Hamiltonian after completing the $(\nu - 1)$-th KAM step

$$H_\nu = N_\nu + P_\nu$$

  where

$$N_\nu = \langle \omega^\nu(\xi), I^\nu \rangle + \sum_{k>\nu} \Omega_k^\nu(\xi)|q_k|^2$$

  and where $\xi = (\xi_1, \cdots \xi_\nu)$ are the parameters
- Renormalization

$$\zeta = (\varepsilon_1^{3/2}\xi_1, \varepsilon_1^2\varepsilon_2^{3/2}\xi_2, \cdots, (\varepsilon_1\varepsilon_2 \ldots \varepsilon_{\nu-1})^2\varepsilon_\nu^{3/2}\xi_\nu)$$

- functions of the frequency parameters will in general be smooth functions of ζ (and therefore satisfy ‘tame’ estimates in ξ).

frequency dependence

- The approximate ν-th tangential frequencies, $k = 1, \dots \nu$ are

$$\omega_k^\nu = \omega_k^\nu(\xi) = \mu_k + \zeta_k + \varepsilon_1^{2/3} f_k^\nu(\zeta)$$

and the ν-th normal frequencies, $k > \nu$ are

$$\Omega_k^\nu = \Omega_k^\nu(\xi) = \mu_k + \varepsilon_1^{2/3} f_k^\nu(\zeta)$$

- The perturbation of the frequencies satisfies

$$\|\partial_{\zeta_j} f_k^\nu(\zeta)\|_{L^\infty(\mathcal{O}^\nu)} \leq Ce^{-\rho|j-k|}$$

- Decompose the perturbation

$$P_\nu = Q_\nu + R_\nu$$

where we count on the part Q_ν for its nonlinear term

$$Q_\nu = (\prod_{j=1}^{\nu} \varepsilon_j^2)\Big(\frac{1}{2}\sum_{k\leq\nu} I_k^2 + \sum_{k>\nu} |q_k|^4\Big)$$

- The variables $(I^{(\nu)}, \theta^{(\nu)})$ are symplectic polar coordinates about a point ξ in action space

$$q_k = \sqrt{(\varepsilon_k^{3/2}\xi_k + \varepsilon_k^2 I_k)}e^{i\theta_k}$$

- The Hamiltonian R_ν contains the rest of the terms

$$R_\nu = \sum_{k\ell\alpha\beta;\,rest} (R_\nu)_{k\ell\alpha\beta}(I^{(\nu)})^{\ell} e^{i(k\cdot\theta^{(\nu)})}\, q^\alpha \overline{q}^\beta$$

Introduce an additional tangential degree of freedom

- Write the $(\nu+1)$-th oscillator as a new degree of freedom

$$z = q_{\nu+1} = \sqrt{(\xi_{\nu+1} + I_{\nu+1})}e^{i\theta_{\nu+1}}$$

- Study the terms of the Hamiltonian R_ν that need to be addressed to regain the normal form

$$R_\nu^N = \sum_{2|\ell|+|\alpha|+|\beta|\leq 4,*} (R_\nu)_{k\ell\alpha\beta}(I^{(\nu)})^\ell e^{i(k\cdot\theta^{(\nu)})} z^{\alpha_1}\overline{z}^{\beta_1}q^{\alpha'}\overline{q}^{\beta'}$$

- The conditions $*$ include $|k| + |\alpha - \beta| > 0$ and in addition that

$$2|\ell| + |\alpha'| + |\beta'| \leq 3\,, \qquad \mathrm{diam}\ \ (\mathrm{supp}(\alpha,\beta)) \leq -\log(\varepsilon_{\nu+1})$$

cohomological equation

- Let the mean value of R_ν be $[R_\nu]$, the cohomological equation is

$$\{N_\nu, F_\nu\} + (R_\nu^N - [R_\nu^N]) = 0$$

- The new Hamiltonian is given by composing with the time-one flow of X_{F_ν}

$$H_{\nu+1} = H_\nu \circ \varphi_{t=1}^{F_\nu}$$

- Renormalizing variables $\xi_{\nu+1} \to \varepsilon_{\nu+1}^{3/2}\xi_{\nu+1}$ and $I_{\nu+1} \to \varepsilon_{\nu+1}^{2} I_{\nu+1}$, we have

$$z = q_{\nu+1} = \sqrt{(\varepsilon_{\nu+1}^{3/2}\xi_{\nu+1} + \varepsilon_{\nu+1}^{2} I_{\nu+1})}e^{i\theta_{\nu+1}}$$

Finally rescale the Hamiltonian $H_{\nu+1} \to \varepsilon_{\nu+1}^{-2} H_{\nu+1}$

Rescaled Hamiltonian

- The rescaled Hamiltonian takes the form

$$\begin{aligned} H_{\nu+1} &= \langle \omega^{(\nu)}, I^{(\nu)} \rangle + \Omega^{\nu}_{\nu+1} I_{\nu+1} + \sum_{k>\nu+1} \Omega^{\nu}_k |q_k|^2 \\ &\quad + (\prod_{j=1}^{\nu+1} \varepsilon_j)^2 \big(\varepsilon_{\nu+1}^{-1/2} \xi_{\nu+1} I_{\nu+1} + \frac{1}{2} \sum_{k=1}^{\nu+1} I_k^2 + \frac{1}{2} \sum_{k>\nu+1} |q_k|^4 \big) \end{aligned}$$

  Thus set $\omega^{(\nu+1)}_{\nu+1}(\xi) = \Omega^{\nu}_{\nu+1} + (\prod_{j=1}^{\nu} \varepsilon_j)^2 \varepsilon_{\nu+1}^{3/2} \xi_{\nu+1}$
- The large-ish constant on the linear term

$$(\prod_{j=1}^{\nu} \varepsilon_j)^2 \varepsilon_{\nu+1}^{3/2} \xi_{\nu+1} I_{\nu+1}$$

  is used in the excision procedure for the next parameter set $\mathcal{O}_{\nu+1}$.
- Choice of small parameter for a convergent scheme $\varepsilon_\nu = \varepsilon_1^{(9/5)^\nu}$

Descriptions of situations in which there are resonant tori

Thank you