On the singular set of the Navier – Stokes equations

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Session on Variational problems
Joint CMS / SMM Meeting
Guanajuato, Mexico
September 22 2006

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Thanks to: NSERC, NSERC-LSI, NSF-FRG
Canada Research Chairs Program

Abstract

- The singular set of a weak solution of Navier – Stokes
- Geometric conditions for the branching of two weak solutions, depending upon their *singular sets*
- The variational equation along an orbit
- Conditions on the growth of Lyapunov exponents for the uniqueness of weak solutions.

Outline

Navier – Stokes equations

The equations of motion of an incompressible viscous fluid

$$\begin{aligned} &\partial_t u + (u \cdot \nabla)u = \Delta u - \nabla p \\ &\nabla \cdot u = 0 \\ &u(x,0) = u_0(x) \qquad \text{initial data} \end{aligned} \tag{1}$$

Space-time domain

$$D = \mathbb{R}^d \qquad (x,t) \in D \times \mathbb{R}^+ := Q$$

Alternatively $D = \mathbb{T}^d$ and

$$(x,t) \in \mathbb{T}^d \times \mathbb{R}^+ = Q$$

A bounded smooth domain $D \subseteq \mathbb{R}^d$; we leave this open.

Weak solutions

The usual definition of weak solutions over $t \in [0, T]$ is that:

1. Integrability conditions

$$\begin{aligned} & u \in L^\infty([0, T]; L^2(D)) \cap L^2([0, T]; H^1(D)) \ , \\ & p \in L^{1+2/d}(Q) \end{aligned} \tag{2}$$

2. The pair (u, p) is a distributional solution of (1)

3. The energy inequality is satisfied

$$\int_D |u(x,t)|^2 \, dx + 2 \int_0^t \int_D |\nabla u(x,s)|^2 \, dxds \leq \int_D |u_0(x)|^2 \, dx \tag{3}$$

Definition (1)

Given a weak solution (u, p) of (1), the singular set $S(u) \subseteq Q$ is the set of space-time points at which $u(x, t)$ is not locally bounded.

That is, $(x_0, t_0) \notin S(u)$ if there is a neighborhood $B_r(x_0, t_0)$ such that

$$u \in L^\infty(B_r(x_0, t_0)) \tag{4}$$

This makes sense due to a theorem of Serrin (1962) which implies that if $(x_0, t_0) \notin S(u)$, then for all k (and with some $0 < \alpha < 1$)

$$\partial_x^k u(x, t) \in C^\alpha(B_{r/2}(x_0, t_0))$$

The condition (4) has been improved by Escauriaza, Seregin and Sveràk (2003) to be $u \in L^\infty L^3(Q_r(x_0, t_0))$, for parabolic cylinders Q_r

Partial regularity

- The singular set $S(u)$ is closed, by definition
- When $d = 3$ the set of singular times $\tau(u) \in \mathbb{R}^+$ has zero $1/2$-Hausdorff dimensional measure

$$\mathcal{H}^{1/2}(\tau(u)) = 0$$

- If $d = 3$ and (u, p) is a suitable weak solution of (1) then the Caffarelli, Kohn & Nirenberg theorem (1982) states that its parabolic one-dimensional Hausdorff measure is zero;

$$\mathcal{P}^1(S(u)) = 0$$

The weak-strong uniqueness principle

There is a second classical theorem of Serrin (1963) which is a uniqueness result in the presence of a strong solution

Theorem (2)

Let (u_1, p_1) *and* (u_2, p_2) *be two weak solutions of* (1) *on the space-time domain* Q*, emanating from the same initial data* $u_0(x)$. *Suppose that in fact* $u_1 \in L^\infty([0,T] \times D) \cap L^2([0,T] : H^{d/2}(D))$. *Then* $u_1 \equiv u_2$ *over the time interval* $[0,T]$.

This states that weak solutions can only branch at singular times. Furthermore, all weak solutions that start from the initial data $u_0(x)$ share the same first singular time T_0.

The regularity condition on u_1 such that the same conclusion holds was recently improved and optimized by Pierre Germain (2006).

First Theorem

The case of $D = \mathbb{T}^d$.

Theorem (3)(A. Biryuk, S. Ibrahim & W. C. (2006))

Let (u_1, p_1) *and* (u_2, p_2) *be two weak solutions of* (1) *on the domain* $[0, T] \times D \subseteq Q$. *Suppose their singular sets satisfy* $S(u_1) \cap S(u_2) = \emptyset$. *Then* $w := u_1 - u_2$ *satisfies*

$$\|w(\cdot, t)\|_2 \leq C_1 \exp(C_2 t)\|w_0(\cdot)\|_2$$

for $0 < t < T$.

The constants C_1, C_2 depend only upon $|u_j|_{L^\infty([0,T]\times D \setminus N(S(u_j)))}$

Corollary (4)

Let (u_1, p_1) *and* (u_2, p_2) *be two weak solutions of* (1) *on the domain* $D \times [0, T] \subseteq Q$ *with the same initial data* $u_0(x)$*. Let* $S(u_1)$ *and* $S(u_2)$ *be their singular sets. Then*

$$S(u_1) \cap S(u_2) \neq \emptyset$$

unless $S(u_1) = S(u_2) = \emptyset$*, in which case* $u_1 \equiv u_2$ *on* $[0, T] \times D$*.*

The significance is that one can only have branching of weak solutions from common space-time singular points.

First theorem - microlocal version

Theorem (5)

Let (u_1, p_1) *and* (u_2, p_2) *be two weak solutions of* (1) *and suppose that*

1. $WF(u_1) \cap WF(u_2) = \emptyset$
2. *For* $z = (x, t) \in S(u_1) \cap S(u_2)$ *then*

$$\zeta = (0, \tau) \notin WF_z(u_1) \cup WF_z(u_2)$$

Then for $0 < t < T$, $w := u_1 - u_2$ *satisfies*

$$\|w(\cdot, t)\|_2 \leq C_1 \exp(C_2 t) \|w_0(\cdot)\|_2$$

In particular, if $w_0 = 0$ *then in this situation* $u_1 \equiv u_2$ *and* $WF(u_j) = \emptyset$

A primer on microlocal analyis

A rapid introduction for the non-microlocal specialists. Let $u(z)$ be a tempered distribution, for $z = (x,t) \in Q$.

- Singular support $S(u)$: The point $z_0 \notin S(u)$ when there exists a cutoff function $\eta \in C_0^\infty$ with $\eta(z_0) = 1$ such that the product $\eta u \in C_0^\infty$
- Equivalently, $z_0 \notin S(u)$ if under Fourier transform,

  $$\widehat{\eta u}(\zeta) \to 0 \text{ for } |\zeta| \to \infty$$
- Wave front set $WF(u)$: The phase space point $(z_0, \zeta^0) \notin WF(u)$ when there exists a cutoff function $\eta \in C_0^\infty$ with $\eta(z_0) = 1$ and a cone Γ containing ζ^0 such that

  $$\widehat{\eta u}(\zeta) \to 0 \text{ for all } |\zeta| \to \infty \text{ with } \zeta \text{ in the cone } \Gamma$$

Theorem: $\mathbb{R}^d$ version

Definition (6)

A system of near singular sets of a weak solution (u, p) is a family of closed sets $S_K(u)$ such that $S(u) = \cap_{K>1} S_K(u)$, and $|u(x,t)| < K$ for $(x,t) \notin S_K(u)$.

Theorem (7)

Let (u_1, p_1) and (u_2, p_2) be two weak solutions of (1) *on $\mathbb{R}^d \times [0, T] \subseteq Q$ with the same initial data $u_0(x)$. Suppose that there is K and near singular sets $S_K(u_1)$, $S_K(u_2)$ which are compact. Then for $0 < t < T$ the difference $w := u_1 - u_2$ satisfies*

$$\|w(\cdot, t)\|_2 \leq C_1 \exp(C_2 t) \|w_0(\cdot)\|_2$$

The issue is one of compactness of the singular set

Compact near singular sets

For $d = 3$, with initial data satisfying

$$\int_{\mathbb{R}^3} |u_0(x)|^2 \, |x| dx < +\infty$$

If (u, p) is a suitable weak solution, the theory of Caffarelli, Kohn and Nirenberg (1982) shows that there do exist compact sets $S_K(u)$

Note on different settings

- On $D = \mathbb{T}^d$ we can adjust

$$\int_{\mathbb{T}^d} u \, dx = 0 \, , \qquad \text{and} \quad p(x,t) \quad \text{periodic}$$

Theorem (8)(our extension of Serrin's *a priori* interior regularity)

Let (u,p) *be a weak solution of* (1) *over* $Q = \mathbb{T}^d \times \mathbb{R}^+$ *with* $\int_{\mathbb{T}^d} u \, dx = 0$ *and* $p(x,t)$ *periodic. If* $(x,t) \notin S(u)$ *then for some* ρ

$$(u,p) \in C^\infty(B_\rho(x,t))$$

Galilean transformations

The issue is over regularity in time. Given any solution (u, p) define a new velocity and pressure field (u', p')

$$
\begin{aligned}
z = (x, t) &\mapsto z' = (x + c(t), t) \\
u(x, t) &\mapsto u' = u(x + c(t), t) - \dot{c}(t) \\
p(x, t) &\mapsto p' = p(x + c(t), t) + \ddot{c}(t) \cdot x
\end{aligned}
$$

Then (u', p') is also a solution of the Navier – Stokes equations.

The function $\dot{c}(t)$ can be quite irregular, indeed only Hölder C^α, $\alpha = 2/(2 + d)$ when the pressure $p \in L^{1+2/d}(Q)$. However boundary conditions place further restrictions on $c(t)$, which lead to the regularity theorem (7).

The variational equations

- Turn to the study of the case of two weak solutions on $\mathbb{T}^d$ for which
$$S(u_1) = S(u_2)$$
and conditions under which we can conclude $u_1 \equiv u_2$.

- Consider one of our weak solutions (u_1, p_1), and linearize the equations about it, writing a variation as $(\delta u_1, \delta p_1) = (v, q)$

$$\partial_t v + (u_1 \cdot \nabla)v + (v \cdot \nabla)u_1 - \Delta v + \nabla q = 0 \qquad (5)$$
$$\nabla \cdot v = 0$$
$$\text{initial data} \quad v(x,0) = v_0(x)$$

Lyapunov exponents

- By (formal) integrations by parts, solutions of the variational equations satisfy a differential identity

$$\partial_t \|w(t)\|_2^2 + 2\|\nabla w\|_2^2 + \int w\Big((u_1\cdot\nabla)w + (w\cdot\nabla)u_1 + \nabla q\Big)\,dx = 0$$

- Denote a solution of this variational identity from $t = T_1$ to $t = T_2$ by $v(\cdot, T_2) = \Phi(T_2, T_1)v(\cdot, T_1)$
- Is there a constant Λ such that

$$\|\Phi(T_2,T_1)v(\cdot,T_1)\|_{L^2} \le e^{\Lambda(T_2-T_1)}\|v(\cdot,T_1)\|_{L^2} \ ?$$

- No. In general $\Lambda = +\infty$

Fourier truncation

- Change the question in the case in which space-time is $\mathbb{T}^d \times (0, T)$. Let Π_N denote projection onto the Fourier modes with wave number $|k| \leq N$.
- Consider solutions $v^N(\cdot, T_2) = \Phi^N(T_2, T_1)v^N(\cdot, T_1)$ of the truncated equations

$$\begin{aligned} &\Pi_N\Big(\partial_t * + (u_1 \cdot \nabla) * + (* \cdot \nabla)u_1\Big)\Pi_N v^N - \Delta\Pi_N v^N + \Pi_N \nabla q^N = 0 \\ &\nabla \cdot \Pi_N v^N = 0 \\ &\text{initial data} \quad \Pi_N v^N(x, 0) = \Pi_N v_0(x) \end{aligned} \tag{6}$$

- Take the best constant Λ_N such that for all $0 \leq T_1 < T_2 \leq T$

$$\|v_N(\cdot, T_2)\|_{L^2} \leq e^{\Lambda_N(T_2 - T_1)}\|v_N(\cdot, T_1)\|_{L^2}$$

Exponents Λ_N

- Of course $\Lambda_N = \Lambda_N(u_1, p_1)$ depends upon the solution. What is the growth of Λ_N in N?
- A universal upper bound

Theorem (9): BCI (2006)

If (u_1, p_1) *is a weak solution of* (1) *then*

$$\Lambda_N \leq C_1 N$$

- But of course it could be less.

Second Theorem

Which conditions imply that weak solutions are unique

Theorem (10): BCI (2006)

Let (u_1, p_1) *and* (u_2, p_2) *be two weak solutions defined on the domain* $\mathbb{T}^d \times [0, T]$, *with the same singular set* $S(u_1) = S(u_2)$. *Let*

$$w := u_2 - u_1$$

- If $w \in C^\omega$ then in fact $w \equiv 0$.
- In case $\Lambda_N = C_1$ bounded, whenever $w \in L^\infty([0, T] : H^1(\mathbb{T}^d))$, then $w \equiv 0$.

Theorem (10)(continued)

- *In case* $\Lambda_N = C_1 \log(N)$, *then* $w \in L^\infty([0,T] : H^{1+\varepsilon}(\mathbb{T}^d))$ *implies* $w \equiv 0$, *for any* ε.
- *When* $\Lambda_N = C_1 N^\beta$, $0 < \beta < 1$ *then* $w \in L^\infty([0,T] : G^\gamma(\mathbb{T}^d))$ *implies* $w \equiv 0$, *where* $\gamma = \gamma(\beta)$.

In any case, the conclusion is that the set of possible different weak solutions is smaller if the Lyapunov exponents of the variational equation grow less rapidly.

Open questions

Theorem (9) is a structural result for the set of weak solutions which arise from a given initial datum $u_0(x)$.

- How to parametrize this set?
- Is the set $S(u)$ a perfect set (meaning that it contains no isolated points)?
- Can we at least show that the first singularity of a weak solution $u(x,t)$ is not isolated?

Thank you